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VALVE SPRING DESIGN - KINEMATICS ANALYSIS

ANALIZA GIBANJA VENTILSKE OPRUGE

Opruga ventila je jednim svojim krajem oslonjena o čvrstu podlogu, dok je drugi kraj izložen periodičnom gibanju koje je određeno profilom brijega bregastog vratila motora. Svaki pojedini harmonik podizaja brijega predstavlja uzbudu koja izaziva vibracije svih točaka opruge. Opruga se pritom ponaša kao vibracijski model s beskonačno mnogo stupnjeva slobode. U literaturi se tvrdi da opruga vibrira gotovo isključivo prvim oblikom. U radu je razmatrana kinematika vibracija opruge, te je pokazano da ona ipak vibrira istovremeno u više oblika. Potvrda za to dobivena je analizom fizikalne interpretacije matematičkih rezultata i analizom publiciranih mjerenja. Također je pokazano kako se na obliku spektra harmonika podizaja brijega može uočiti utjecaj viših harmonika na rezultirajući oblik vibriranja opruge.

1. INTRODUCTION

The valve spring is retained at one end and controlled by a cam at the other end. A method of harmonic analysis can be used to replace the lift function by series of sinus harmonics. Each harmonic is one disturbance function that induces vibrations of all particles of a spring. The spring possesses infinite number of degrees of freedom that is infinite number of resonant frequencies. The motion of particles of a spring is described by *wave equation.*

In this paper author tries to explain some theoretical unclearness, found in a accessible literature, in connection with the response factor (HUSSMANN) and dynamic displacement. In opposite to frequent opinions (HUSSMANN, STRAUBEL), it will be shown that two or more modes may consist in a spring at the same time. Special attention has been paid to the physical interpretation and analysis of mathematical results.

2. LONGITUDINAL VIBRATION OF A BAR

2.1. The Differential Equation Of Motion.

When considering vibration of helical spring, it can be replaced by a bar with continuously distributed mass, elasticity and dumping, where A is a cross sectional area of a bar, l is its length, ρ is density if material, E is Young's modulus of elasticity and c is a dumping coefficient. Free vibration of a spring corresponds to those of a bar whose both ends are fixed, while at forced vibration only one end is retained and the other end performs harmonic motion. The displacement $u(x,t)$ along the bar (*Fig. 1*) will be a function of both, the position x and the time t.

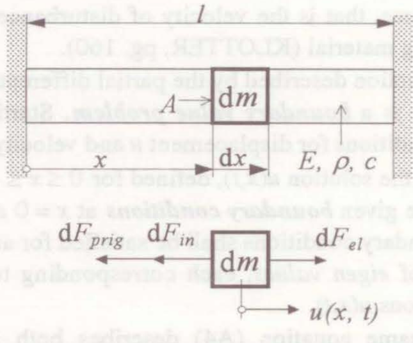


***Figure 1* Longitudinal vibration of a bar: Forces acting on an element *dm*.**

The elastic strain force dF_{el} is:

$$dF_{el} = A \cdot d\sigma ,$$

$$\sigma_x = E \cdot \varepsilon_x ,\ \varepsilon_x = \frac{\partial u\,(x,\,t)}{\partial x}$$

$$d\sigma_x = E \cdot d\varepsilon_x = E \cdot \left(\frac{\partial \varepsilon_x}{\partial x} dx + \frac{\partial \varepsilon_x}{\partial t} dt\right)$$

where: $d\sigma_x$ - differential of normal stress σ_x in cross sectional area A if x increases for dx, ε_x - unit elongation.

While considering the whole bar in the same time $t = const.$, we obtain:

$$d\sigma_x = E \cdot \frac{\partial \varepsilon_x}{\partial x} dx = E \cdot \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\right) dx = E \cdot \frac{\partial^2 u}{\partial x^2} dx .$$

$$dF_{el} = A \cdot E \cdot \frac{\partial^2 u}{\partial x^2} dx . \qquad (A1)$$

The inertia force dF_{in} is:

$$dF_{in} = dm\frac{\partial^2 u}{\partial t^2} = A \cdot \rho \cdot \frac{\partial^2 u}{\partial t^2}dx. \quad (A2)$$

The dumping force dF_{prig} is equal to the product: *dumping coefficient* × *velocity*[1] $= c \cdot (\partial u/\partial t)$. If the dumping force F_{prig} is continuously distributed over the length of the bar: $F_{prig}/l = (\partial u/\partial t) \cdot c/l$, differential of force, that corresponds to the element dx, is:

$$dF_{prig} = \frac{F_{prig}}{l} \cdot dx = \frac{c}{l} \cdot \frac{\partial u}{\partial t}dx. \quad (A3)$$

Substituting (A1, A2 and A3) in the equation for equilibrium: $dF_{in} + dF_{prig} = dF_{el}$, we obtain the differential equation of motion:

$$\frac{\partial^2 u}{\partial t^2} + 2b\frac{\partial u}{\partial t} = a^2\frac{\partial^2 u}{\partial x^2}, \quad b = \frac{c}{2m}, \quad a^2 = \frac{E}{\rho} \quad (A4)$$

This is a wave equation with a dumping term represented by $2b(\partial u/\partial t)$, where ***a*** is the *sound velocity* in solid structures, that is the velocity of disturbance moving in a spring material (KLOTTER, pg. 160).

The motion described by the partial differential equation (A4) is a ***boundary value problem***. Starting from initial conditions for displacement u and velocity $\partial u/\partial t$ at time t=0, the solution $u(x,t)$, defined for $0 \le x \le 1$, has to satisfy the given ***boundary conditions*** at x = 0 and $x = l$. This boundary conditions shall be satisfied for an infinite number of *eigen values*, each corresponding to one of the solutions $u(x,t)$.

The same equation (A4) describes both *free* and *forced* vibration, varying only boundary conditions. At *free* vibration they are: $u(0,t) = 0$, $u(l,t) = 0$, and at *forced* vibration: $u(0, t)=0$, $u(l, t)=R\cdot\sin(\omega t+\delta)$, where R is the amplitude of exciting harmonic, ω is its angular frequency and δ its phase angle.

Solving *free vibration of the bar* we obtain the *eigen values*, which are equal to *resonant frequencies*. The *displacement u* will be found by solving *forced vibration*.

2.2. Free Vibration

Equation (A4) can be solved by separation of variables. Therefore, the function u can be written as a product:

$$u(x, t) = X(x) \cdot T(t), \quad (A5)$$

where X is the function of the position x only, and T is the function of the time t only. From the expression for u we calculate $\partial u/\partial t, \partial^2 u/\partial t^2, \partial u^2/\partial x^2$ and substituting them in (A4) we get:

$$\frac{T''}{T} + 2b\frac{T'}{T} = a^2\frac{X''}{X}, \quad (A6)$$

The terms on the left are functions of time only, these on the right functions of positions only. So both sides must be equal to the same constant $-\nu^2$. We obtain the homogeneous case of the system of two linear differential equations of the 2[nd] order, with constant coefficients:

$$T'' + 2bT' + \nu^2 T = 0 \quad (A7.1)$$

$$X'' + \frac{\nu^2}{a^2}X = 0 \quad (A7.2)$$

The solutions of equations (A7) can be obtained only for certain values of ν^2. The function X is called *eigen function*, and ν^2 is *eigen value* of a boundary problem.

Solution of equation (A7.2) should be searched for in the form: $X = e^{rx}$. We calculate: $X' = re^{rx}$, $X'' = r^2e^{rx}$, and by substituting it in the equation (A7.2) we obtain the characteristic equation: $r^2 + \nu^2/a^2 = 0$, its roots are: $r_{1,2} = \pm i \cdot \nu/a$. The general solution: $X = C_1e^{r_1x} + C_2e^{r_2x}$, can be transformed into: $X = A_1\cos\frac{\nu}{a}x + A_2\sin\frac{\nu}{a}x$, by means of Euler's formula[2] and by substituting the solutions r_1 and r_2.

Corresponding to the ***first boundary condition***: $X = A_2\sin\frac{\nu}{a}x \quad u(0, t) = 0$, it is: $u = X \cdot T = 0$. This shall be satisfied if: $X = 0$, is followed by: $A_1 = 0$, so the general solution is: $X = A_2\sin\frac{\nu}{a}x$

By applying the ***second boundary condition***: $u(l, t) = 0$ we get: $\sin\frac{\nu}{a}l = 0$, with the conclusion that there has to exist the eigen value ν or *resonant frequency* (*frequency of normal mode of vibration*):

$$\nu = \lambda \cdot \pi \cdot \frac{a}{l}, \text{ rad/s} \quad (A8)$$

where:

$\lambda = 1, 2, 3, \ldots$ - order of normal mode,

$a = \sqrt{\frac{E}{\rho}}$, m/s - sound velocity in a bar.

From (A8) we obtain: $\frac{\nu}{a} = \lambda \cdot \pi \cdot \frac{1}{l}$, so the final expression for *eigen function X*:

$$X = A_2\sin\left(\lambda \cdot \pi \cdot \frac{x}{l}\right). \quad (A9)$$

Solution of equation (A7.1) should be also searched for in the form: $T = e^{rt}$. The roots of the characteristic equation:

$$r^2 + 2br + v^2 = 0 ,$$

are: $r_{1,2} = -b \pm i\sqrt{v^2 - b^2}$, while $v^2 - b^2 > 0$. The general solution: $T = D_1 e^{r_1 t} + D_2 e^{r_2 t}$, can be transformed to:

$$T = e^{-bt}\left(D_3 \cos\left(\sqrt{v^2 - b^2 t}\right) + \right.$$

$$\left. + D_4 \sin\left(\sqrt{v^2 - b^2 t}\right)\right) . \quad \text{(A10)}$$

The displacement u can be obtained by substituting the expression (A9) for X and expression (A10) for T into (A5):

$$u = X \cdot T = e^{-bt}\left(K_1 \cos\left(\sqrt{v^2 - b^2 t}\right) + \right.$$

$$\left. + K_2 \sin\left(\sqrt{v^2 - b^2 t}\right)\right) \cdot \sin\left(\lambda \cdot \pi \cdot \frac{x}{l}\right) . \quad \text{(A11)}$$

The ***resonant angular frequency of dumped vibration*** is, extracting from this equation:

$$v_b = \sqrt{v^2 - b^2} . \quad \text{(A12)}$$

Constants K_1 and K_2 can be calculated by applying the initial kinematics conditions: $u(x, 0) = f(x)$, $\dot{u}(x, 0) = g(x)$.

For the solution of the differential equation (A7) we assumed: *1.* that vibration of the bar could be treated independently for each of the normal modes and its eigen value, and *2.* if the bar was vibrating at one of its normal modes, all its points were vibrating with the same simple harmonic motion: $u = X \cdot \sin(\omega t)$.

2.3. Forced Vibration.

We shall solve the equation (A4) by using the complex method. The forced vibration is the sum of free vibration and excitation. In a dumped system, however, the free vibration is dumped out rapidly and only steady-state oscillation remains, with an excitation frequency ω. The complex solution is:

$$u(x, t) = X(x) \cdot T(t) = X \cdot e^{i(\omega t + \delta)} . \quad \text{(A13)}$$

Now we obtain:

$$\frac{\partial^2 u}{\partial x^2} = X'' \cdot e^{i(\omega t + \delta)} ,$$

$$\frac{\partial u}{\partial t} = X \cdot i\omega \cdot e^{i(\omega t + \delta)} = i\omega u , \text{[3]}$$

$$\frac{\partial^2 u}{\partial t^2} = X \cdot i^2 \omega^2 \cdot e^{i(\omega t + \delta)} = -\omega^2 u .$$

By substituting this into u (A4), we get[4]:

$$X'' - \left(-\frac{\omega^2}{a^2} + i \cdot 2b\frac{\omega}{a^2}\right)X = X'' - \gamma^2 X = 0 , \quad \text{(A14)}$$

where[5]:

$$\gamma = \sqrt{-\frac{\omega^2}{a^2} + i \cdot 2b\frac{\omega}{a^2}} = \alpha + i\beta , \ \alpha \approx \frac{b}{a} , \ \beta \approx \frac{\omega}{a} . \quad \text{(A15)}$$

The characteristic equation: $r^2 - \gamma^2 = 0$, has its complex roots: $r_{1,2} = \pm\gamma$. The general solution is: $X = C_1 e^{\gamma x} + C_2 e^{-\gamma x}$. The ***first boundary condition***: $u(0, t) = X \cdot T = 0$ produces: $X(0) = 0$, followed by: $C_2 = -C_1$. Thus the general solution is:

$$X = C_1 e^{\gamma x} - C_1 e^{-\gamma x} = 2C_1 \operatorname{sh}(\gamma x) . \quad \text{(A16)}$$

In the ***second boundary condition***: $u(l, t) = (R \cdot \sin(\omega t + \delta))$, the term: $\sin(\omega t + \delta)$ is the function of time t only. So, we can say that R is a function of position x only: $X(l) = R$. It must be equal to (A16) and then we get: $C_1 = \dfrac{R}{2\operatorname{sh}(\gamma l)}$. Substituting C_1 in (A16) and then substituting X in (A13) we obtain the complex dynamic displacement u: $u(x, t) = X \cdot T = R \cdot \dfrac{\operatorname{sh}(\gamma x)}{\operatorname{sh}(\gamma l)} \cdot e^{i(\omega t + \delta)} = R \cdot V_u^* \cdot e^{i(\omega t + \delta)}$, where V_u^* is the complex response factor. By applying the exponential formula of the complex number[6], general trigonometric and hyperbolic formulas[7], we get:

$$V_u^* = \frac{\operatorname{sh}(\gamma x)}{\operatorname{sh}(\gamma l)} =$$

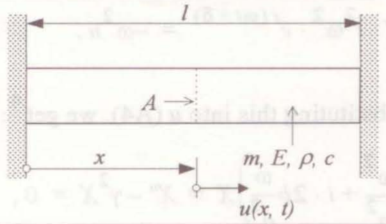
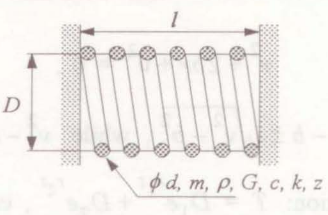
$$\sqrt{\frac{\operatorname{sh}^2(\alpha x) + \sin^2(\beta x)}{\operatorname{sh}^2(\alpha l) + \sin^2(\beta l)}} \cdot e^{i\varphi} = V_u \cdot e^{i} . \quad \text{(A17)}$$

We obtain the dynamic displacement u as the imaginary part of the complex displacement:

$$u(x, t) = R \cdot V_u \cdot \sin(\omega t + \delta + \varphi) . \quad \text{(A18)}$$

where V_u is the response factor and φ is the phase angle between the excitation and the dynamic displacement u.

Table 1 **Analogy of vibration of a bar and a helical spring**

Prismatic Bar	Helical spring
	
$E = \frac{\sigma}{\varepsilon} = \frac{\sigma}{\frac{\Delta l}{l}} = \frac{\sigma}{\frac{\partial u}{\partial x}}$	$E = \frac{\sigma}{\varepsilon} = \frac{\frac{F}{A'}}{\frac{\Delta l}{l}} = \frac{l}{A'} \cdot \frac{F}{\Delta l} = \frac{l}{A'} \cdot k = \frac{lGd^4}{2D^5 \pi z}$ $A' = \frac{D^2 \pi}{4}, \ k = \frac{Gd^4}{8D^3 z}$
$a^2 = \frac{E}{\rho}$ $a = \sqrt{\frac{E}{\rho}}$	$a^2 = \frac{E}{\rho'}, \ \rho' = \frac{m}{V'} = \frac{\pi z}{Dl} \cdot \rho, \ m = D\pi z \cdot \frac{d^2 \pi}{4} \rho$ $a = \frac{ld}{\pi D^2 z} \cdot \sqrt{\frac{G}{2\rho}} = l \cdot \sqrt{\frac{k}{m}}$
$\nu_b = \sqrt{\nu^2 - b^2} = \nu \sqrt{1 - \zeta^2}, \ \nu_1 = \sqrt{\frac{k}{m}}$	$\nu_b = \sqrt{\nu^2 - b^2} = \nu \sqrt{1 - \left(\frac{\zeta}{\pi}\right)^2}, \ \nu_1 = \pi \sqrt{\frac{k}{m}}$
$\zeta = \frac{c}{c_{krit}} = \frac{c}{2\sqrt{km}}, \ c_{krit} = 2\sqrt{km}$	

A, m^2 - cross sectional area; ***a***, m/s - sound velocity; ***b***, 1/s - dumping; ***c***, kg/s - dumping coefficient; ***E***, Pa - modulus of elasticity; ***F***, N - force; ***G***, Pa - modulus of stiffness; ***k***, N/m - spring rate; ***m***, kg - active mass; *z* -number of active coils; ρ, kg/m^3 - density of material; σ, Pa - normal stress; ν/ν_b, rad/s - natural angular frequency (ν_1 - the lowest) of damped/undamped system; ζ - fraction of critical dumping; ***A', E, V'***, ρ' - symbols corresponding to helical spring treated as a bar with diameter *D* and length *l*.

They are equal to:

$$V_u = \sqrt{\frac{\mathrm{sh}^2(\alpha x) + \sin^2(\beta x)}{\mathrm{sh}^2(\alpha l) + \sin^2(\beta l)}}, \quad \text{(A19)}$$

$$\varphi = \arctan\frac{\tan(\beta x)}{\tanh(\alpha x)} - \arctan\frac{\tan(\beta l)}{\tanh(\alpha l)} \quad \text{(A20)}$$

3. SPRING VIBRATION

If we define all the different terms, the formulas for a prismatic bar can also be applied for a helical spring. Instead of the distance *x* we shall use the relative distance *x/l*, to exclude the influence of the length *l* of the spring.

3.1. Dynamic Displacement u_μ.

At the moving end of a spring the disturbance function is represented by the harmonic of μ^{th} order: $u_{\mu(x=l)} = R_\mu \cdot \sin(\mu\omega_{BV} t + \delta_\mu)$, where δ_μ is the phase angle between the harmonic and the cam rotation angle φ_{BV} By substituting: $\omega t = \mu\varphi_{BV}$ into expression (A18), using $\psi = -\varphi$, we obtain the dynamic displacement u_μ (*Fig.* 2) of the μ^{th} order harmonic of the cam lift function, at the camshaft speed n_{BV}, at distance *x*, as the function of the cam rotation angle φ_{BV} [8]:

$$\boxed{u_\mu = R_\mu \cdot V_{u,\mu} \cdot \sin(\mu\varphi_{BV} + \delta_\mu - \psi_{u,\mu})}, \ \mathrm{m} \quad \text{(A21)}$$

where: R_μ, m - amplitude of the μ^{th} harmonic of the cam lift;
$V_{u\mu}$ - response factor of the μ^{th} harmonic;
φ_{BV}, rad - cam rotation angle;
δ_μ, rad - phase angle of the μ^{th} harmonic,
$\psi_{u,\mu}$, rad - phase angle between the excitation harmonic and the dynamic displacement *u*.

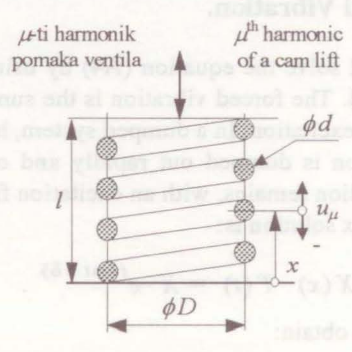


***Figure 2* The considered point is at the distance *x* from fixed end of the spring, the dynamic displacement is *u*.**

3.2. Response factor $V_{u,\mu}$ and phase angle $\psi_{u,\mu}$.

The response factor $V_{u\mu}$ of the harmonic of the μ^{th} order (*Fig. 3.*) at the camshaft speed n_{BV}, at distance x, corresponding to (A19), and in the form suitable for the computer[10]:

$$V_{u,\mu} = \mathrm{sign}\left(\sin\left(\beta l\left(1-\frac{x}{l}\right)\right)\right)\cdot\sqrt{\frac{\mathrm{sh}^2\left(\alpha l\frac{x}{l}\right)+\sin^2\left(\beta l\cdot\frac{x}{l}\right)}{\mathrm{sh}^2(\alpha l)+\sin^2(\beta l)}}$$

if: $\frac{x}{l} = 1$, or if: $n_{BV} = 0$, then: (A22)

$$V_{u,\mu} = \sqrt{\frac{\mathrm{sh}^2\left(\alpha l\frac{x}{l}\right)+\sin^2\left(\beta l\frac{x}{l}\right)}{\mathrm{sh}^2(\alpha l)+\sin^2(\beta l)}}$$

where: x, m - the distance of considered point from fixed end of a spring; l, m - length of a spring.

The terms α and β are:

$$\alpha l \approx \frac{b}{a}\cdot l = b\cdot\frac{\pi}{\nu_1} = b\cdot\sqrt{\frac{m}{k}} = $$

$$= b\cdot\frac{\pi D^2 z}{d}\cdot\sqrt{\frac{2\rho}{G}} = \zeta \qquad \text{(A23)}$$

$$b = \frac{c}{2m} = \zeta\cdot\frac{a}{l} = \zeta\cdot\sqrt{\frac{k}{m}} =$$

$$= \zeta\cdot\frac{\nu_1}{\pi} = \zeta\frac{d}{\pi D^2 z}\cdot\sqrt{\frac{G}{2\rho}}, \qquad \text{(A24)}$$

where: **a**, m/s - sound velocity (velocity of disturbing wave moving) in a spring treated as a prismatic bar; **b**, s^{-1} - dumping; **c**, kg/s - dumping coefficient; **d, D**, m - wire diameter, mean coil diameter; **G**, Pa - modulus of stiffness; **k**, N/m - spring rate; **l**, m - length of a built in spring (valve at rest); **m**, kg - active mass of a spring; **z** - number of active coils; ν_1, rad/s - lowest natural angular frequency of a spring; ρ, kg/m³ - density of spring material; ζ - fraction of critical dumping.

$$\beta l \approx \frac{\omega_\mu}{a}\cdot l = \mu\omega_{BV}\cdot\frac{l}{a} =$$

$$= \mu\omega_{BV}\frac{\pi}{\nu_1} = \mu\omega_{BV}\frac{\pi D^2 z}{d}\sqrt{\frac{2\rho}{G}} \qquad \text{(A25)}$$

where: ω_μ, rad/s - angular frequency of harmonic of the μ^{th} order of the cam lift function; μ - order of harmonic; ω_{BV}, rad/s - angular frequency of camshaft; **n_{BV}**, s^{-1} - camshaft speed (frequency); $\omega_{BV} = 2\pi n_{BV}$.

The response factor $V_{u,\mu}$, as defined in the expression (A22), is a function of: dumping b, the lowest angular natural frequency ν, the order μ of the harmonic, the camshaft speed, as well as the position x of observed point of a spring:

$$V_{u,\mu} = V_{u,\mu}\left(\alpha(b,\nu_1),\ \beta(\mu,\omega_{BV},\nu_1),\ \frac{x}{l}\right)$$

The phase angle $\psi_{u,\mu}$ between the excitation μ^{th} harmonic and the dynamic displacement u, in the form suitable for the computer[9], is:

if $\varphi_{u,\mu} < 0$ then $\psi_{u,\mu} = -\varphi_{u,\mu}$,

else if $\varphi_{u,\mu} > 0$ then $\psi_{u,\mu} = \pi - \varphi_{u,\mu}$. (A26)

where $\varphi_{u,\mu}$ is obtained from (A20):

$$\varphi_{u,\mu} = \arctan\frac{\tan\left(\beta l\cdot\frac{x}{l}\right)}{\tanh\left(\alpha l\cdot\frac{x}{l}\right)} - \arctan\frac{\tan(\beta l)}{\tanh(\alpha l)} \qquad \text{(A27)}$$

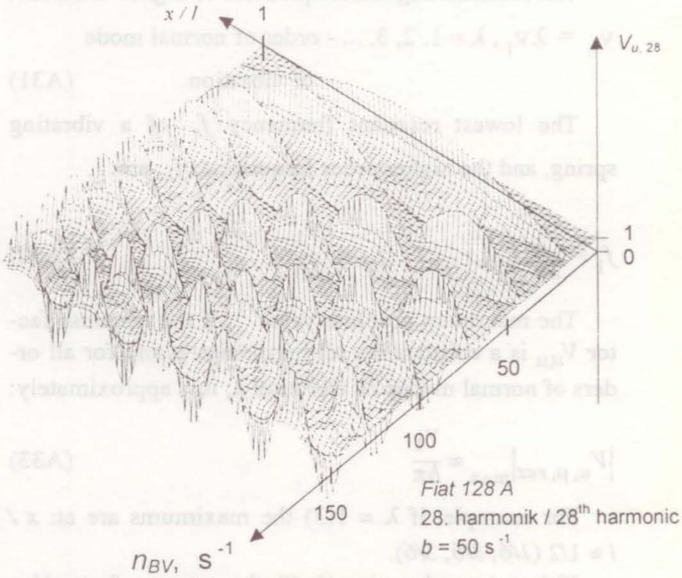


Figure 3 **Response factor $V_{u,\mu}$ of harmonic of μ^{th} order.**

3.3. Resonance.

Accordingly to the expression (A22) the response factor $V_{u\mu}$ reaches its maximum if $\sin(\beta l) = 0$. From there we get the ***resonant condition***:

$$\beta_{rez} l = \lambda\pi, \qquad (\lambda = 1, 2, 3, \ldots) \qquad \text{(A28)}$$

According to (A15) is: $\beta = \omega_\mu / a$, resulting with: $\beta_{rez} = \omega_{\mu,rez}/a$. However, the *angular frequency* ω_μ of μ^{th} harmonic (that is the *angular frequency* of the amplitude vector R_μ) is μ-times bigger than the cam angular frequency ω_{BV}, so it will be the same at the *reso-*

nant frequency: $\omega_{\mu, rez} = \mu \cdot \omega_{BV, rez}$. By substituting this into expression (A28) we get the *resonant condition*: $(\omega_{\mu, rez}/a) \cdot l = \lambda\pi$, and the resulting *resonant* angular frequency of the μ^{th} harmonic is: $\omega_{\mu, rez} = \lambda\pi \cdot a/l$. If we substitute the equation for a, the ***resonant angular frequency of a cam*** is:

$$\boxed{\omega_{BV, rez}} = \frac{\lambda\pi}{\mu} \cdot \frac{a}{l} = \frac{\lambda\pi}{\mu} \cdot \sqrt{\frac{k}{m}} =$$

$$= \frac{\lambda}{\mu} \cdot \frac{d}{D^2 z} \cdot \sqrt{\frac{G}{2\rho}} = \boxed{\frac{\lambda \nu_1}{\mu}}, \text{ rad / s} \qquad (A29)$$

where ν_1 is the *lowest angular frequency of a spring (undumped*[10]*)*:

$$\boxed{\nu_1 = \pi \cdot \frac{a}{l} = \pi \cdot \sqrt{\frac{k}{m}} = \frac{d}{D^2 z} \cdot \sqrt{\frac{G}{2\rho}}}, \text{ rad / s} \qquad (A30)$$

The resonant angular frequencies of higher order are:

$\nu_\lambda = \lambda \nu_1$, $\lambda = 1, 2, 3, \ldots$ - order of normal mode of vibration. (A31)

The lowest resonant frequency f_1 of a vibrating spring, and the higher order frequencies f_λ, are:

$$f_1 = \frac{\nu_1}{2\pi}, \text{ s}^{-1} \qquad f_\lambda = \frac{\nu_1}{2\pi}, \text{ s}^{-1} \qquad (A32)$$

The maximum absolute value[11] of the response factor $V_{u,\mu}$ is a constant for all harmonics μ and for all orders of normal modes of vibration λ, it is approximately:

$$|V_{u, \mu, rez}|_{max} \approx \frac{\nu_1}{b\pi} \qquad (A33)$$

For example, if $\lambda = 1$(*3*) the maximums are at: $x/l \approx 1/2$ (*1/6, 3/6, 5/6*).

If there is no dumping (b=0), the response factor $V_{u,\mu}$ (A22), and the dynamic displacement u_μ (A21), tend to infinity.

The resonant phase angle $\psi_{u, rez}$ of the dynamic displacement u is equal for all excitation harmonics μ and it can be obtained by expression (A26):

if $\varphi_{u, rez} < 0$ then $\psi_{u, rez} = -\varphi_{u, rez}$,

else if $\varphi_{u, rez} > 0$ then $\psi_{u, rez} = \pi - \varphi_{u, rez}$.

The value $\varphi_{u, rez}$ can be get by substituting the resonant condition $\beta_{rez} l = \lambda\pi$ into (A27), that is:

$$\varphi_{u, rez} = \arctan \frac{\tan\left(\lambda\pi \cdot \frac{x}{l}\right)}{\tanh\left(\alpha l \cdot \frac{x}{l}\right)} \qquad (A34)$$

For example, at 1[st] mode of vibration (λ=1, $\omega_{BV, rez}$ accordingly expression (A35)), using small dumping b, the phase angle ψ is approximately equal $\pi/2$ over the whole length of a spring, except for starting value *zero* at the moving end of a spring.

3.4. The Interpretation of ν.

Presuming that we deal with a cam profile designed to produce a disturbance single sinus wave during each revolution of the cam rotation. The angular frequency of the wave is equal to the angular frequency of a cam: $\omega_1 = \omega_{BV}$. Assuming the frequency of the cam to be equal to the lowest natural angular frequency: $\omega_{BV} = \nu_1$. The spring then performs the resonant vibration of the 1[st] normal mode (λ = 1). At twice the cam frequency, the spring vibration would occur at double frequency $\omega_{BV} = \nu_2$ and with the 2[nd] normal mode (λ = 2).

In general, if a condition of resonance is satisfied, the angular frequency of a cam is:

$$\omega_{BV, rez} = 2\pi n_{BV, rez} = \frac{\lambda}{\mu} \nu_1, \qquad (A35)$$

while the angular frequency of the disturbance harmonic is: $\omega_\mu = \mu\, \omega_1 = \omega_{BV, rez}$, the response factor $V_{u\mu}$ reaches its absolute maximum, and the order of the normal mode of vibration is λ.

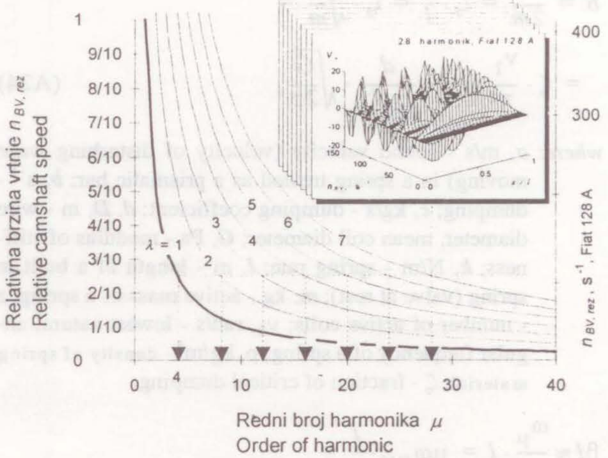


Figure 4 **The resonant camshaft speed depending upon the order of the cam lift harmonic for different orders of modes of vibration λ.**

At *resonant frequency of a cam* $n_{BV, rez}$ *(Fig. 4)* the frequency of the μ^{th} harmonic is equal to any of the natural frequency of a spring:

$$f_{BV, rez} = n_{BV, rez} = \frac{\lambda}{\mu} f_\lambda = \frac{\lambda}{\mu} \frac{\nu_1}{2\pi}. \qquad (A36)$$

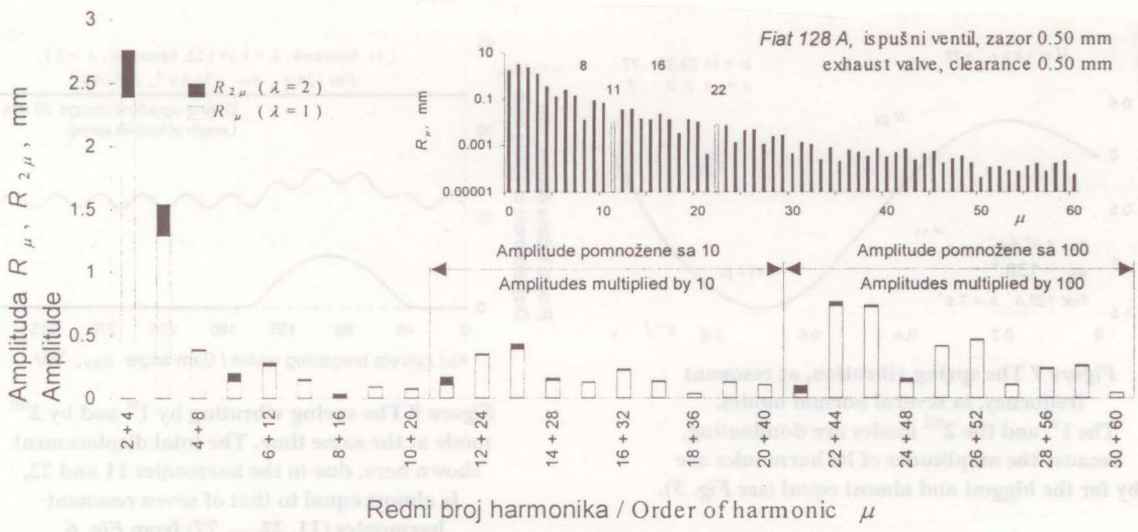


Figure 5 **The line spectrum of the Fourier series of the cam is a useful tool to estimate which of pairs of (μ^{th} and $2\mu^{th}$) harmonics have approximately equal amplitudes.**

3.5. Displacement.

Total sum of displacements u of series consisting of μ harmonics is:

$$u(x,t) = \sum_{\mu} u_{\mu}(x,t) \qquad \text{(A37)}$$

How many terms should a series of harmonics have if calculating the displacement? Examining the example of analysis of a cam profile of the car engine *Fiat 128 A* (see the details at the end of the paper).

According to the graph (*Fig. 5*) the amplitudes of 8^{th} and 16^{th} harmonic are almost equal as well as those of the 11^{th} and 22^{nd} harmonics. The camshaft speed[12] of $n_{BV} = 37{,}4\ s^{-1}$ is a resonant speed of the 11^{th}, 22^{nd}, 33^{rd}, ... harmonics. As shown in *Fig.* 6, for rapid decay of the response factor $V_{u,\mu}$ it is enough that the camshaft speed differs just slightly from the resonant speed. So the contribution of the nonresonant harmonics is irrelevant. This shows that the total sum of dynamic displacements of a series of *seven* resonant harmonics (*Fig.7*), is almost identical to the sum of displacements of only the first *two* resonant harmonics[13].

It is obvious then, that the *series of harmonics* should have enough terms to include the 1^{st} and the 2^{nd} normal mode of vibration (μ^{th} harmonic by $\lambda=1$ and $2\mu^{th}$ harmonic by $\lambda=2$) in the considered range of the camshaft speed. HAFNER, from KHD, calculated the displacement u using the series of 32 harmonics. For a quick analysis it will be enough to observe the resonant frequencies (equations A29 and A36) in a range up to the highest engine speed and to include in the calculation only the μ^{th} and the $2\mu^{th}$ harmonics.

But, if the frequency of the μ^{th} harmonic: $\omega_{\mu} = \mu \cdot \omega_{BV}$, by the 1^{st} mode, coincides with any of the resonant frequencies ν_{λ}, and if its amplitude is much larger than the $2\mu^{th}$ amplitude, then the motion will be almost completely determined by only the 1^{st} mode (*Fig. 8*).

Amplitude U_{μ} of the dynamic displacement u, at the position x and at the camshaft speed n_{BV} can be obtained from (A21) by substituting $\sin(\mu\varphi_{BV} + \delta_{\mu} - \psi_{u,\mu}) = 1$:

$$U_{\mu} = R_{\mu} \cdot V_{u,\mu} \qquad \text{(A38)}$$

According to (A22) at the fixed end of a spring is

$x = 0$: $\quad V_{u,\mu} = 0$, followed by: $U_{\mu} = 0$,

while at the moving end it is

$x = l$: $\quad V_{u,\mu} = 1$, followed by: $U_{\mu} = R_{\mu}$.

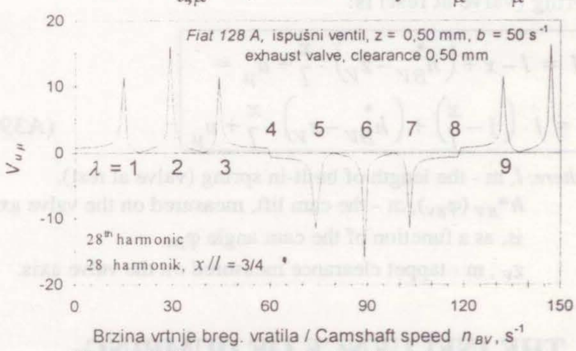


Figure 6 **The response factor $V_{u,\mu}$ decreases rapidly if the camshaft leaves its resonant speed values (this graph is the cross sectional area of surface $V_{u,\mu}$ from *Fig. 3*).**

Consequently, the amplitude of the displacement *at the end of a spring*, that is controlled by the cam, is equal to the amplitude of the excitation's harmonic.

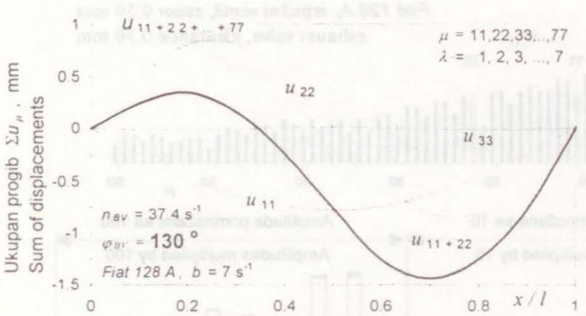


Figure 7 **The spring vibration, at resonant frequency, in several normal modes. The 1st and the 2nd modes are dominating, because the amplitudes of its harmonics are by far the biggest and almost equal (see *Fig. 5*).**

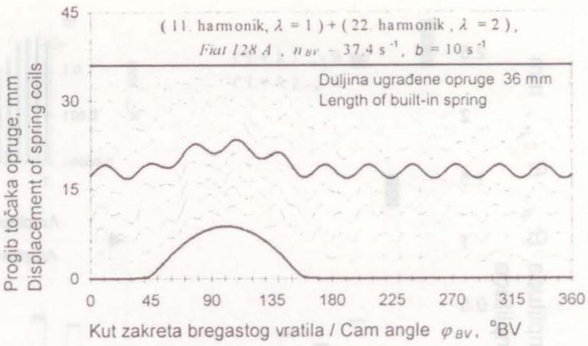


Figure 9 **The spring vibrating by 1st and by 2nd mode at the same time. The total displacement shown here, due to the harmonics 11 and 22, is almost equal to that of seven resonant harmonics (11, 22, ..., 77) from *Fig. 6*.**

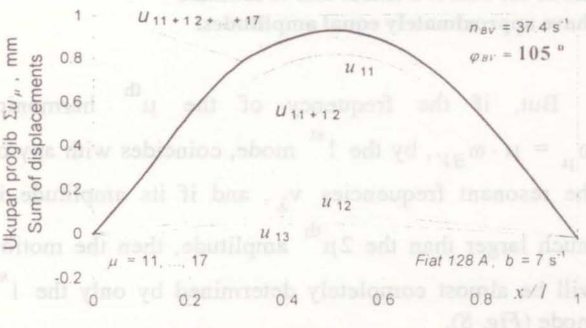


Figure 8 **The total displacement of series of 7 harmonics. The motion is almost completely determined by the 11th harmonic which is in a resonance (with 1st mode).**

The total displacement H of the observed point of a spring, produced by static compression due to the cam lift, and by vibration due to the μ^{th} harmonic, at the camshaft speed n_{BV}, measured from the fixed end of a spring (valve at rest) is:

$$H = l - x + \left(h^*_{BV} - z_V\right) \cdot \frac{x}{l} + u_\mu = l \cdot \left(1 - \frac{x}{l}\right) + \left(h^*_{BV} - z_V\right) \cdot \frac{x}{l} + u_\mu \quad \text{(A39)}$$

where: l, m - the length of built-in spring (valve at rest),
h^*_{BV} (φ_{BV}), m - the cam lift, measured on the valve axis, as a function of the cam angle φ_{BV},
z_V, m - tappet clearance measured on the valve axis.

4. THE INFLUENCE OF DUMPING AND ENDCOILS

The calculation of dynamic displacement u, is not possible without knowing the value of dumping. The data from the literature: According to HUSSMANN[14] the dumping is: from $b \approx 7 \div 12$ ($\zeta \approx 0{,}0085 \div 0{,}024$), up to 20 s^{-1} ($\zeta = 0{,}061$). HUNDAL calculated with the value $\zeta = 0{,}001$ ($b = 0{,}82$ s^{-1}). By the slightly modified theory HAFNER used the constant ratio of dumping and camshaft speed, pointing out the necessity of experiments to find it.

When considering the *measuring* of dumping HUSSMANN (p. 34, 35) has mentioned two possibilities: determining of logarithmic decrement[15] Δ and 2. measuring of the *resonant amplitude* of displacement, that is the determining the response factor $V_{u,\mu,rez}$ by the already known sinus wave excitation with the amplitude R_μ, and the calculation of dumping b from the expression:

$$U_{\mu, rez, \max} = R_\mu \cdot \left|V_{u,\mu,rez}\right|_{\max} \approx R_\mu \cdot \frac{\nu}{b\pi} \quad \text{(A40)}$$

HUNDAL's (reference 7., p. 6, 7) calculation of the fraction of critical dumping ζ was also based on experiments, but he considers several cycles in a row. For the single degree of freedom system he calculated the logarithmic decrement from the formula:

$$\Delta = \frac{1}{2\pi n} \ln \frac{N_0}{N_n} \quad \text{(A41)}$$

where: N_0 - the amplitude of the first cycle, N_n the amplitude of the n^{th} cycle following it, $2\pi/\nu$, s - the cycle period.

Another circumstance has also significant influence on the results of the calculation. Valve spring endcoils are *squared and ground* flat. The intercoil gap at the spring endcoils is gradually reduced to zero, resulting in a *loss of active coils* and *active mass*. The significance of this change is reported by ROSKILLY and FERAN.

5. CONCLUSION

The end of the spring ($x=l$), which is controlled by the cam, performs the motion that can be replaced by the series of sinus harmonics by method of harmonic analysis. Therefore, the motion of the endpoint of the spring is equal to the total sum of all harmonics. If the spring vibrated only by the 1^{st} mode (HUSSMANN, STRAUBEL), then somewhere near that end an area, where the power of higher order harmonics would disappear, would have to exist.

Probably it does not often happen that at resonant frequencies the amplitudes of the μ^{th} and $2\mu^{th}$ harmonic are approximately equal. But only in such case it is relatively simple to notice that it vibrates by the 1^{st} and the 2^{nd} mode consisting in the spring concurrently. HUSSMANN's experiments, where the simultaneous vibration of a spring by the 1^{st} and 2^{nd} mode was recorded, were analysed. It was found out, that these cases were of the same type like described here.

It can be asserted that, at the resonant frequency, the spring vibrates simultaneously by all modes and due to all excitation harmonics. As the amplitudes decrease with the order of harmonics, in most of cases the 1^{st} mode of vibration dominates and followed by the 1^{st} and the 2^{nd} mode simultaneously. When analysing vibration, the line spectrum of amplitude series (*Fig. 5*) should be used to estimate which of the pairs of (μ^{th} and $2\mu^{th}$) harmonics, with the resonant frequencies (A36) in the engine speed range, have approximately equal amplitudes. The resonant vibrations due to these harmonics will be dominated by the 1^{st} and the 2^{nd} mode simultaneously. All other vibrations will almost completely follow the 1^{st} mode of motion.

6. EXAMPLE

Given: Four cycle combustion engine Fiat 128 A, external valve spring: the main coil diameter $D = \phi\ 27{,}4$; the wire diameter $d = \phi\ 3{,}8$; the free length of a spring 54 mm; the built-in length $l = 36$ mm; the number of active coils $z = 4{,}5$; the spring material: steel; $\rho = 7850$ kg/m^3; $G = 8{,}3\times10^{10}$ Pa; dumping $b = 20$ s^{-1}; the amplitude of 9^{th} harmonic $R_9 = 0{,}088$ mm; the maximum of valve lift $(h^*_{BV} - z_V) = 8{,}75$ mm.

The amplitude of displacement of the vibrating spring has to be calculated. The resonance excitation is the 9^{th} harmonic that has a very large amplitude (see *Fig. 5*).

Results:

The lowest natural frequency of a spring (A30):

$$\nu_1 = \frac{d}{D^2 z}\sqrt{\frac{G}{2\rho}} =$$

$$= \frac{3{,}8\cdot 10^{-3}}{(27{,}4\cdot 10^{-3})^2\cdot 4{,}5}\cdot\sqrt{\frac{8{,}3\cdot 10^{10}}{2\cdot 7850}} = 2586\frac{\text{rad}}{s}.$$

The resonance camshaft speed for the 9^{th} harmonic (A36):

$$n_{BV,rez} = \frac{\lambda\cdot\nu_1}{2\pi\mu} = \frac{1\cdot 2586}{2\pi\cdot 9} = 45{,}7\ \text{s}^{-1}$$

Maximum of the response factor (A33):

$$|V_{u,\mu,rez}|_{max} \approx \frac{\nu_1}{b\pi} = \frac{2586}{20\cdot\pi} = 41{,}2\,.$$

The resonance amplitude of displacement (9^{th} harm.) (A40):

$$U_{9,max} = R_9|V_{u,\mu,rez}|_{max} = 0{,}088\cdot 41{,}2 = 3{,}6\ \text{mm}.$$

The 1^{st} normal mode of vibration (λ=1) appears at the camshaft speed $n_{BV} = 45{,}7$ s^{-1} and at the relative distance x/l=0.5 (the middle coil); the 2^{nd} mode (λ=2) appears at $n_{BV} = 91{,}5$ s^{-1} (5488 min^{-1}) and at the distance x/l=1/3; ... (The crankshaft speed at the maximum of power is 100 s^{-1} or 6000 min^{-1}). The resonant amplitude $U_{9,max} = 3{,}6$ mm is the same at all normal modes of vibration.

The results of approximate formulas for α and β, and the influence of **dumping** on the natural frequency ν_1 of a spring, at the camshaft speed $n_{BV} = 45{,}7$ s^{-1} and with dumping $b = 7$ s^{-1}:

$\alpha = 0{,}23620026$ $\quad\beta = 87{,}26678$ *exact formula* (A15a),
$\quad 0{,}23620034$ $\quad 87{,}26646$ *appr. formula* (A15),
$\quad 3{,}4\times10^{-5}$ %, $\quad 3{,}7\times10^{-4}$ % *the difference.*

$\nu_1 = 2586{,}18$ rad / s *without* dumping (A30),
$\nu_{1,b} = 2586{,}16$ rad / s *with* dumping (A12),
the difference $= 7{,}7\times10^{-4}$ %.

If the dumping is $b = 20$ s^{-1} the differences of α, β and $\nu_{1,b}$ are equal: $3{,}0\times10^{-3}$ % of the exact value.

By means of approximate formulas it is very simple to express the correlation (A23) between the dumping b and the *fraction of critical dumping* ζ. However, this does not influence the results of the calculations.

SUMMARY

The paper considers the surge of a spring, retained at one end and controlled by a cam at the other end. The periodic lift function of the cam was approximately replaced by the Fourier polynom. The forced displacement amplitudes were determined for each harmonic of this polynom. The theory of forced longi-

tudinal vibrations of a bar with continuously distributed mass, elasticity and dumping was applied to simulate the motion of spring coils. It was found out that two or more modes may consist concurrently in the spring. This was obtained by analyzing mathematical formulas and by comparing computed and published measured results. The line spectrum of the Fourier series was introduced as a useful tool for estimating the influence of the high order harmonics on the spring surge.

NOTES:

1. Relative velocity of element dm and its neighboring element.
2. Euler's formula for complex numbers:
$$e^{\alpha+i\beta} = e^{\alpha}\cdot e^{i\beta} = e^{\alpha}\cdot(\cos\beta + i\sin\beta)\ ;\ e^{\alpha} = \left|e^{\alpha+i\beta}\right|.$$
3. The rotation of complex number for an angle of $\pi/2$ is equal to its multiplication by imaginary unit i:
$$\frac{\partial u}{\partial t} = \omega\cdot X\cdot\cos(\omega t+\delta) = \omega\cdot X\cdot\sin(\omega t+\delta+\pi/2) =$$
$$= i\cdot\omega\cdot X\cdot\sin(\omega t+\delta) = i\cdot\omega\cdot u$$
4. The variables are separate again: X is the function of x only.
5. Exact formulas for α and β (the excitation frequency is ω_μ):
$$\alpha = \frac{\omega_\mu}{a}\cdot\sqrt{\frac{1}{2}\cdot\left(\sqrt{1+\left(\frac{2b}{\omega_\mu}\right)^2}-1\right)},$$
$$\beta = \frac{\omega_\mu}{a}\cdot\sqrt{\frac{1}{2}\cdot\left(\sqrt{1+\left(\frac{2b}{\omega_\mu}\right)^2}+1\right)} \quad \text{(A15a)}.$$
If the dumping b is as small as it is for springs, it can be neglected as it is done in formulas (A15). See the example at the end of the paper.
6. $\mathrm{sh}(\gamma x) = \mathrm{sh}(\alpha x + i\cdot\beta x) = \mathrm{sh}(\alpha x)\cdot\cos(\beta x) +$
$$+\, i\cdot\mathrm{ch}(\alpha x)\cdot\sin(\beta x) = A + i\cdot B = \sqrt{A^2+B^2}\cdot e^{i\,\mathrm{arctg}\frac{B}{A}}$$
7. $\sin^2\alpha + \cos^2\alpha = 1$, $\mathrm{ch}^2\alpha - \mathrm{sh}^2\alpha = 1$
8. All the significant terms are defined by equations with frames.
9. If: $\tan x < 0$, then computer gives the result: $x < 0$, but never $x > \pi/2$. To control the *sign* of the displacement u, the function *signum* was used in the expression for $V_{u,\mu}$ (A22), instead of the phase angle $\varphi_{u,\mu}$.
10. The influence of dumping on resonant frequency could be neglected, see example at the end.
11. The response factor $V_{u,\mu,rez}$ at resonant camshaft speed can be obtained by substituting the expression (A28) in equation (A22). As the dumping b is very small, it can be written: $\alpha\, l \approx 0$, followed by: $\mathrm{sh}(\alpha l) \approx \alpha l = \frac{b\pi}{\nu}$,
$\mathrm{sh}^2\left(\alpha l\cdot\frac{x}{l}\right) \approx 0$, and finally: $V_{u,\mu,rez} \approx \frac{\nu}{b\pi}\cdot\sin\left(\lambda\pi\cdot\frac{x}{l}\right)$.
We get the extremes if: $\sin\left(\lambda\pi\cdot\frac{x}{l}\right) = 1$, and that is at the distance x: $\frac{x}{l} = \frac{2K-1}{2\lambda}$, $K = 1, 2, 3, \ldots, \lambda$.
12. It is from expression (A36):
$$n_{BV,rez} = \frac{\lambda}{\mu}\cdot\frac{\nu_1}{2\pi} = \frac{1}{11}\cdot\frac{2586}{2\pi} = \frac{2}{22}\cdot\frac{2586}{2\pi} =$$
$$= \frac{3}{33}\cdot\frac{2586}{2\pi} = \ldots = 37{,}4\ \mathrm{s}^{-1}$$
13. As shown in the *Fig.* 5, the amplitudes R_μ of the higher order harmonics are even smaller. In general, the amplitudes of the μ^{th} harmonics (λ=1) are much greater than those of the $2\mu^{th}$ harmonics (λ=2), which are again much greater than those of the $3\mu^{th}$ harmonics (λ=3), and so on. So the contribution of the normal modes of high order (λ) is mainly not as big as observed in an unusual case like in *Fig.* 5.
14. HUSSMANN made his experiments using valve springs from aircraft engines.
15. The *logarithmic decrement* Δ is the natural logarithm of the ratio of the amplitudes N_1 and N_2 of two successive cycles of dumped free vibration:
$$\Delta = \ln\frac{N_1}{N_2} = \text{(single degree-of-freedom system)} =$$
$$= \zeta\nu T = \frac{2\pi\zeta}{\sqrt{1-\zeta^2}},$$
where: ν, rad/s - the natural angular frequency of vibration; νT, rad - the cycle period ($\nu\, T = 2\pi$); T, s - the cycle period of dumped vibration.
The logarithmic decrement shows the decay of amplitudes of dumped vibration. Without dumping, there would be no decay. If $\Delta = 0$ then: $\ln(N_1/N_2) = 0 \Rightarrow N_1/N_2 = 1$

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