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Science in Traffic
Original Scientific Paper
U.D.C. 656.1/.5
Accepted: Oct. 22, 1977
Approved: Jan. 30, 1998

AN APPROXIMATE MODEL FOR FLEET SIZING AND REDISTRIBUTION

SUMMARY

To allocate a fleet of vehicles to a given number of locations and to redistribute free vehicles are important control problems. In this paper we use the results for multi-location inventory models to develop an approximate solution for the combined fleet sizing and redistribution problem. For a defined reward structure we investigate:

a) some properties of the optimal redistribution and allocation decision respectively;

b) the concavity of the expected one-period reward;

c) the advantages gained from co-operation of locations over independent locations.

Since a model with discrete time is used, the proposed solution is an approximation. The two-location model is investigated in more detail.

1. INTRODUCTION

Consider a transportation company that maintains a fleet of vehicles which are distributed among a given number N of locations. Let the infinite planning horizon be divided into periods. During each period the vehicles have to meet a random transportation demand. The demand in a given location can be served by vehicles located in the terminal or by vehicles transferred from other terminals. At the end of a period the unserved demand is lost. Furthermore, we assume for location i the following cost structure:

- a profit g_i per served demand unit,
- a penalty p_i per unserved demand unit,
- transfer cost c_{ij} per vehicle transferred to location $j \neq i$,
- operating and maintenance costs h_i per vehicle located to i.

The decision problem is to find such a number of vehicles for each terminal and a rule for vehicle transfer which maximise the expected discounted reward over the planning horizon, i.e. we are looking for the optimal fleet size as well as for an optimal redistribution policy.

The search for optimal fleet allocation decisions (AD) and redistribution decisions (RD) is an important problem in the control of transportation systems. In the past, attempts have been made to solve this problem using mathematical programming. For instance, [DUHA97] uses ideas from the queuing theory and mathematical inventory theory to investigate a special structured centre-terminal system with no direct flow of transportation equipment between terminals. In this paper we propose an approximate solution for the formulated fleet sizing and redistribution problem by the use of results from inventory theory and from Markovian decision theory. For this reason we model the above described situation as a special multi-location inventory model with contingents and redistribution as it is introduced and investigated in [KÖCH90]. In the next section we will give the formal description of the problem. The static or one-period model is investigated in Section 3. We will see that the solution of the static model is a solution for the dynamic model as well. In Section 4 we consider the two-location case in detail. Finally, in Section 5 we make some comments on further research of the problem.

2. THE MULTI-LOCATION MODEL

The multi-location system comprises N interconnected locations, $N \geq 2$. Let the planning horizon be divided into periods $t \in \mathbf{T} = \{1,2, \dots\}$. During period $t \in \mathbf{T}$ a transportation demand occurs in accordance with a non-negative random vector $\underline{s} = (\underline{s}_1, \underline{s}_2, \dots, \underline{s}_N)$ where random variable $\underline{s}_i$ describes the demand in location i, $i = 1(1)N$. We denote the distribution function and the density for the random vector $\underline{s}$, by F( ) and $\mathbf{f}$( ) respectively. Let $E(\underline{s}) = \mu = (\mu_1, \mu_2 .. \mu_N)$ exist with $0 < \mu_i < \infty$ for $i = 1(1)$ N. The demand is assumed to be stationary and independent over time, and later in Section 4 across locations as well. The transportation equipment (vehicles, containers, ...) are located at the locations to meet this arriving transportation demand.

The demand is measured in transportation units, for instance cubic metres or quintals. Let n_i denote the number of transportation equipment, let us say vehicles, allocated to location i, i=1(1)N. We assume (cp. [DUHA97] that all vehicles are identical in the sense that they can serve at the same moment exactly one demand unit. Furthermore, let $t_{average}$ and t_{period} denote the average number of time units to serve a demand unit or the number of time units of a period respectively. The travel time from location i to location j is assumed to be insignificant with respect to the size of a period, i, j=1(1)N. Finally we assume that at the beginning of a period all n_i vehicles are available at location i, i=1(1)N. Thus we can compute the transportation capacity available at location i during a period as $a_i = n_i \times t_{period} / t_{average}$, i= 1(1)N. To simplify further investigations we replace the integer variable n_i by the non-negative real variable a_i, i=1(1)N. For the given transportation capacity a_i we can approximate the corresponding number of transportation equipment through

$$n_i = \lceil a_i \times t_{average} / t_{period} \rceil, \qquad (2.1)$$

where $\lceil x \rceil$ denotes the smallest integer greater or equal x. If $a_i \geq 0$ denotes the transportation capacity which is scheduled for location i, i=1(1)N, then $A = a_1 + a_2 \ldots + a_N$ denotes the total capacity of the system, and the non-negative vector $\mathbf{a} = (a_1, a_2, .., a_N)$ describes the allocation of transportation capacity A to the N locations. Throughout the paper we will call such a vector **a** an allocation decision (AD). At the end of a period, after demand is observed, it is possible to redistribute the not used transportation capacity by a redistribution decision (RD). Let $\mathbf{y} = \mathbf{a} - \mathbf{s}$ denote the vector of net capacities if demand realisation **s** is observed, i.e. y represents the vector of pre-redistribution capacity levels. The RD $\mathbf{b} = (b_{ij})_{i,j=1..N}$ is a transfer plan, i.e. b_{ij} denotes the amount of transportation capacity transferred from location i to location j. It is obvious that for the given **y** the set of admissible transfer plan **B(y)** is defined as

$$\mathbf{B}(\mathbf{y}) = \left\{ \mathbf{b} = (b_{ij}) : b_{ij} \geq 0, \quad \sum_{i=1}^{N} b_{ij} = (y_i)^+, \, i, j = 1..N \right\},$$

where $(x)^+ = \max(0, x)$

Transfer occurs immediately by cost c_{ij} for one unit of transportation capacity transferred from location i to location j. After realisation of the RD the unsatisfied demand is lost. In accordance with Section 1 we assume the following gain and cost structure:

(a) Each demand unit served by a capacity unit of location i brings a profit g_i, i=1(1)N.

(b) Each demand unit in location i, which remains unserved after the realisation of the RD, causes a penalty p_i, i=1(1)N.

(c) Each capacity unit, which is transferred from location i to location j, causes transfer costs c_{ij}, i,j=1(1)N.

(d) For each transportation capacity unit allocated to location i we have costs k_i, i=1(1)N.

It remains to clarify the relation between costs h_i and k_i for the given i. For this we use the equation $n_i h_i = a_i k_i$ or

$$k_i = n_i / a_i \, h_i \qquad (2.2)$$

From equ. (2.1) we have

$$n_i - 1 < a_i \times t_{average} / t_{period} \leq n_i$$

or

$$t_{average} / t_{period} \leq n_i / a_i < t_{average} / t_{period} + 1 / a_i \,.$$

If we put these inequalities into equ. (2.2.) we get

$$0 \leq k_i - h_i \times t_{average} / t_{period} < h_i / a_i \,.$$

It means that for sufficiently great capacity values a_i the cost parameter k_i can be approximated by

$$h_i \times t_{average} / t_{period} \,.$$

Thus we put

$$k_i = h_i \times t_{average} / t_{period} \quad \text{for } i = 1(1)N \,. \qquad (2.3)$$

According to the cost and profit parameters we assume that the following condition holds:

$$0 = c_{ii} \leq k_i, g_i, p_i, c_{ij} < \infty, \quad i,j = 1..N, i \neq j. \qquad (2.4)$$

Rewards in period t will be discounted by the factor α^{t-1}, $0 \leq \alpha \leq 1$. For $0 < \alpha < 1$ the problem is to find an optimal control policy, i.e. a sequence of AD's and RD's that maximises the expected discounted reward over an infinite planning horizon. If $\alpha = 0$ we have the one-period or the static problem. For $\alpha = 1$ we use the average criterion, where we have to find a policy that maximises the expected average reward over an infinite horizon.

Remark 2.1:

In fact, the modelling of the initial problem as an inventory model leads to an approximate solution of the fleet sizing and redistribution problem. This is the consequence of the following restrictive assumptions that are made more or less visible in the description of the model:

(1) The demand is continuous.

This assumption may be of less influence on the solution if the size of a demand unit is small relative to the order of the optimal capacity.

(2) The transportation capacities are theoretical available capacities.

From $a_i = n_i \times t_{period} / t_{average}$ it follows that we assign an average capacity to any given number of vehicles. Furthermore, that capacity can serve the corresponding demand only in the case when the demand realisation is concentrated to the beginning of a period. At the end we underestimate the optimal solution. The estimation error will be less if $t_{average}$ is approximately equal to t_{period}.

(3) The RD is realised immediately.

If the travel times between locations are small relative to t_{period} then this assumption is not restrictive.

(4) The once allocated capacity is available again and again at the beginning of a period.

It means that in practice all vehicles will return to their locations up to the beginning of the next period.

In the following section we investigate the static problem.

3. THE STATIC PROBLEM

In this section we investigate the one-period multi-terminal model with redistribution. The investigation is based on results that are received in [KÖCH82] and [KÖCH90]. We use the notions of the previous section to specify the one-period gain function. For that purpose we define function **G(a,s,b)** which represents the total gain for AD **a**, demand realisation **s**, and RD **b**. From the description of the model we get

$$G(\mathbf{a}, \mathbf{s}, \mathbf{b})=\sum_{i=1}^{N} -k_i\, a_i + g(\mathbf{a}, \mathbf{s}, \mathbf{b}) \quad , \tag{3.1}$$

where $g(\mathbf{a}, \mathbf{s}, \mathbf{b})$ denotes the profit minus penalties and transfer costs in the multi-location model for AD **a**, demand realisation **s**, and RD **b**. Then

$$G(\mathbf{a}, \mathbf{s})= \max_{b\in B(a,s)} G(\mathbf{a}, \mathbf{s}, \mathbf{b}) \quad , \tag{3.2}$$

represents the total gain for **a** and **s** under optimal transhipments. Finally, function

$$G(\mathbf{a})= \int_{\{s\geq 0\}} G(\mathbf{a}, \mathbf{s})\, f(\mathbf{s})\, ds \quad , \tag{3.3}$$

denotes the maximal expected gain for AD **a** and the corresponding optimal RD. Function G( ) represents the one-period gain function. The problem is to define an optimal AD, i.e. an

$$\mathbf{a}^* \geq 0=(0,0,\ldots,0): \quad G(\mathbf{a}^*)=\max_{a\geq 0} G(\mathbf{a}) \quad , \tag{3.4}$$

However, we have no analytical tractable expressions for function G( ). The real reason is connected with the RD. To compute for the given pair **(a, s)** function $G(\mathbf{a}, \mathbf{s})$, defined in (3.2), we need to solve a linear transportation problem with excess and shortage. For such problems closed form solutions do not exist.

The first step to overcome these difficulties is to get structural properties of the optimal RD. For the given **a** and **s** the N locations will be divided into two disjoint sets $\mathbf{I}^+ = \{ i = 1..N: a_i > s_i \}$ - the set of locations with positive net capacity after realisation of the demand, and $\mathbf{I}^- = \{ i = 1..N: a_i < s_i \}$ - the set of locations with corresponding negative net capacity. The locations in set I^+ as well as in set $\mathbf{I}^-$ cause costs. To decrease these costs we organise a transfer of not used capacity from locations in set $\mathbf{I}^+$ to locations in set $\mathbf{I}^-$. To avoid non-economic capacity transfers we introduce some additional conditions on the cost and profit parameters. We distinguish two cases.

Case 1: Transfer from $\mathbf{I}^+$ into $\mathbf{I}^-$ should be efficient, i.e. a transfer should decrease the total costs. For this we assume the condition "**E**fficiency of **T**ransfers"

$$\text{(ET)} \quad g_j + p_j > c_{ij} \ \text{ for i, j= 1(1)N} ,$$

i.e. for the transfer or one capacity unit from $i \in \mathbf{I}^+$ to $j \in \mathbf{I}^-$ we have to pay c_{ij} transfer costs, but we gain $g_j + p_j$ in location j.

Case 2: Transfer from $\mathbf{I}^+$ into $\{1, 2, \ldots, N\} \setminus \mathbf{I}^-$ should be inefficient, i.e. a transfer should increase the total costs. This leads to the condition "**R**elative **I**ndependence of the locations"

$$\text{(RI)} \quad c_{ij} + p_j > p_i \ \text{ for i, j= 1(1)N, } i\neq j.$$

Condition (RI) means that a transfer between locations with unused capacity is unprofitable.

A third assumption is the "**S**hortest **W**ay" condition

$$\text{(SW)} \quad c_{ir} + c_{rj} > c_{ij},\ \text{i, j,r= 1(1)N ,} i\neq j\neq r.$$

Condition (SW) expresses that it is cheaper to transfer directly than via another location.

Finally, if for a given location j' there is a location i' such that $k_{j'} \geq k_{i'} + c_{i'j'}$, we can "close" location j' because of the allocation of transportation capacity in j' is not cheaper than to allocate that capacity in location i' and to transfer it from i' to j'. Thus we assume the "**R**eal **A**llocation" condition

$$\text{(RA)} \quad k_i + c_{ij} > k_j \ \text{ for i, j= 1(1)N.}$$

The consequence of conditions (ET), (RI), and (SW) is that in the optimal RD unreasonable transfers will not occur. The following lemma, which can be proved as in [KÖCH75], summarises corresponding properties.

Lemma 3.1.

Let conditions (ET), (RI), and (SW) be fulfilled. Then for the optimal RD $\mathbf{b}^* = (\mathbf{b}_{ij}^*)$ holds:

(1) $b_{ik}^* \, b_{kj}^* = 0$ for i, j, k= 1(1)N, $i\neq j\neq k$.

(2) $b_{ij}^* = 0$ for i= 1(1)N and $j\notin \mathbf{I}^-$, $i\neq j$.

(3) $\sum_{i\in \mathbf{I}^+} b_{ij}^* \leq s_j - a_j$ for $j \notin \mathbf{I}^-$

(4) $\sum_{i\in \mathbf{I}^+} \sum_{i\in \mathbf{I}^-} b_{ij}^* = \min\left[\sum_{i\in \mathbf{I}^+} (s_i - a_i) \, ; \, \sum_{i\in \mathbf{I}^-} (a_j - s_j) \right]$

Properties (1) to (4) for the optimal RD can be interpreted as follows.

Property (1): No location can both initiate and receive transfers.

Property (2): Transfers between locations with unused capacities are not optimal. This is a consequence of condition (RI).

Property (3): It is not optimal to transfer to a location $j \in \mathbf{I}^-$, i.e. to a location with shortage capacities, more than the amount of the shortage.

Property (4): The total amount of transfers is equal to the minimum of the total unused capacities and of the total shortages.

Using these properties we get for the function g(**a**, **s**): = g(**a**, **s**, **b***) with g(**a**, **s**, **b**) from (3.1) the following expression:

$$g(\mathbf{a},\mathbf{s})=\sum_{i=1}^{N} g_i \min(a_i, s_i)+\sum_{i\in\mathbf{I}^+}\sum_{j\in\mathbf{I}^-} b_{ij}^{*}\, g_j -$$

$$-\sum_{i\in\mathbf{I}^-} p_j\left(s_j-a_j-\sum_{i\in\mathbf{I}^+} b_{ij}^{*}\right)-\sum_{i\in\mathbf{I}^+}\sum_{j\in\mathbf{I}^-} c_{ij}\, b_{ij}^{*}=$$

$$=\sum_{i=1}^{N} g_i \min(a_i, s_i)-\sum_{i\in\mathbf{I}^-} p_j(s_j-a_j)+$$

$$+\sum_{i\in\mathbf{I}^+}\sum_{j\in\mathbf{I}^-}(g_j+p_j-c_{ij})b_{ij}^{*} \tag{3.5}$$

Now it follows from the equations (3.1), (3.2), (3.3) and (3.5) that

$$G(\mathbf{a})=\sum_{i=1}^{N}\left\{(g_i+p_i)\int_0^{a_i}[1-F_i(s_i)]\,ds_i-(k_j\, a_i+p_i\,\mu_i)\right\}$$

$$+\,C(\mathbf{a}) \tag{3.6}$$

where $F_i(\,)$ denotes the marginal distribution function of $\underline{s}_i$, i = 1(1)N.

Function C( ) is defined by

$$C(\mathbf{a})=\int_{\{s\geq 0\}}\sum_{i\in\mathbf{I}^+}\sum_{j\in\mathbf{I}^-} c_{ij}\, b^{*}_{ij}\, f(\mathbf{s})\, d\mathbf{s} \tag{3.7}$$

with

$$C_{ij}=g_j+p_j-c_{ij}\ ,\ i,j=1(1)N\ ,\ i\neq j\,. \tag{3.8}$$

Thus function C(**a**) from equ. (3,7) represents the maximal expected gain from the co-operation of initially N independent locations with given capacity a_i for location i, i=1(1) N. Now the interpretation of equ. (3.6) becomes obvious: The expected reward for the given AD **a** and optimal RD **b*** is equal to the total expected reward earned from independent locations with corresponding capacities **plus** the expected gain C(**a**) from transfers.

It can be shown (see e.g. [KÖCH75]), that g(**a**, **s**) is a concave function of **a** and **s**. Since the density function f( ) is non-negative it follows that G(**a**) from (3.3) or (3.6) is a concave function of a_1 to a_N. On the basis of this concavity property we can prove in the same way as in [KÖCH75] the following important result.

Theorem 3.1.

Let for the static N-location model with redistribution the assumptions (ET), (RI), (SW), (RA), and condition (2,4) be fulfilled. Then it holds:

(1) The optimal AD **a*** = ($a_1, a_2, \ldots a_N$) can be uniquely defined from the system of equations

$$\delta\, G(\mathbf{a})/\delta\, a_i=0\ ,\ i=1(1)N.$$

(2) The repeated application of AD **a*** and RD **b*** leads to an optimal control for the dynamic model with or without discounting.

In the following section we consider the two-location model in more detail.

4. THE 2-LOCATION CASE

For N=2 locations the optimal transfer decision **b*** can be easily defined. From the properties of **b*** we get the following:

(a) Let $s_i < a_i$ for i=1,2.

Then we have $\mathbf{I}^+ = \{1, 2\}$, $\mathbf{I}^- = \varnothing$, and $b^{*}_{ii} = a_i - s_i$, $b^{*}_{i\,3\text{-}i} = b^{*}_{3\text{-}i\,i} = 0$, i = 1,2.

(b) Let $s_i > a_i$ for i = 1,2.

Now we have $\mathbf{I}^- = \{1,2\}$, $\mathbf{I}^+ = \varnothing$, and $b^{*}_{ij} = a_i$, $b^{*}_{i\,3\text{-}i} = b^{*}_{3\text{-}i\,i} = 0$, i= 1,2. We notice that both in case (a) and in case (b) the optimal RD dictates no proper redistribution.

(c) Let $s_i > a_i$, $s_{3\text{-}i} < a_{3\text{-}i}$, $s_1 + s_2 \leq a_1 + a_2$ for given i=1,2.

It holds that $\mathbf{I}^+ = \{3\text{-}1\}$, $\mathbf{I}^- = \{i\}$, and $b^{*}_{ii} = a_i$, $b^{*}_{3\text{-}i\,3\text{-}i} = s_{3\text{-}i}$,

$b^{*}_{3\text{-}ii} = \min\{a_{3\text{-}i} - s_{3\text{-}1};\ s_i - a_i\} = s_i - a_i$, i = 1,2.

(d) Let $s_i > a_i$, $s_{3\text{-}i} < a_{3\text{-}i}$, $s_1 + s_2 > a_1 + a_2$ for the given i = 1, 2.

It follows that $\mathbf{I}^+ = \{3\text{-}i\}$, $\mathbf{I}^- = \{i\}$, and $b^{*}_{ii} = a_i$, $b^{*}_{3\text{-}i\,3\text{-}i} = s_{3\text{-}i}$,

$b^{*}_{3\text{-}i\,i} = \min\{a_{3\text{-}i} - s_{3\text{-}i};\ s_i - a_i\} = a_{3\text{-}i} - s_{3\text{-}i}$, i = 1, 2.

With these results we can compute function C(**a**) = C(a_1, a_2). To simplify this computation we assume:

The random variables $\underline{s_1}$ and $\underline{s_2}$ are independent random variables with distribution function $F_i(\,)$, continuous density $f_i(\,)$, and $F_i(0) = 0$ for i = 1,2…

Then from equ. (3.7) and (3.8) we get

$$C(a_1,a_2)=$$

$$=\sum_{i=1}^{2} C_{i,3-1}\left[\int_0^{a_i}\int_{a_{3-i}}^{a_1+a_2-s_i}(s_{3-i}-a_{3-i})f_{3-i}(s_{3-i})\,ds_{3-i}\,f_i(s_i)\,ds_i\ +\right.$$

$$\left.+\int_0^{a_i}\int_{a_1+a_2-s_i}^{\infty}(a_i-s_i)f_{3-i}(s_{3-i})\,ds_{3-i}\,f_i(s_i)\,ds_i\right]=$$

$$=\sum_{i=1}^{2} C_{i,3-i}\int_0^{a_i}F_i(y)[1-F_{3-i}(a_i+a_2-y)]\,dy$$

Consequently we have from equ. (3.6) that

$$G(a_1,a_2)=-\sum_{i=1}^{2} k_i\, a_i\ -\sum_{i=1}^{2} p_i\,\mu_i\ +$$

$$+\sum_{i=1}^{2}(g_i+p_i)\int_0^{a_i}[1-F_i(y)]\,dy+ \tag{4.1}$$

$$+\sum_{i=1}^{2} C_{i,3-i}\int_0^{a_i}F_i(y)[1-F_{3-i}(a_1+a_2-y)]\,dy,\ a_1,a_2\geq 0$$

The optimal AD **a*** = (a^{*}_1, a^{*}_2) is a solution of the system $\delta G/\delta a_1 = 0$, i=1, 2. It follows from (4.1) that

$$\frac{\delta G}{\delta a_i}=-k_i+[1-F_i(a_i)](g_i+p_i)+$$

$$+\,C_{i,3-i}\,F_i(a_i)\,[1-F_{3-i}(a_{3-i})]-$$

$$-\sum_{i=1}^{2} C_{i,3-i} \int_0^{a_i} F_i(y)\, f_{3-i}(a_1+a_2-y)\, dy, i=1,2. \quad (4.2)$$

From (4.2) we derive the inequalities $g_i + p_i > k_i$, i =1, 2, as a necessary condition for $\mathbf{a}^* > (0,0)$, i.e. the optimal fleet size is positive if the profit for a served demand unit plus the penalty for an unserved demand unit is greater than the costs for a transportation capacity unit.

To get a better insight into the changes involved by the transition from independent work to co-operation of locations we consider a simple example with two locations.

Example 4.1.

We assume the following data: N = 2; $F_i(s) = 1 - \exp(-s/\mu_i)$, $s \geq 0$, $\mu_i>0$, i = 1, 2; and

i	k_i	g_i	p_i	c_{i1}	c_{i2}	μ_i
1	7	12	3		1	50
2	7	15	5	3		80

With these data we conclude from equ.(3.8) that $C_{12} = 19$ and $C_{21} = 12$. For function $G(a_1, a_2)$, given in equ. (4.1), we get

$$G(a_1,a_2) = G^\circ(a_1,a_2) + C(a_1,a_2)$$

with

$$G^\circ(a_1,a_2) =$$

$$=\sum_{i=1}^{2}\left[g_i\,\mu_i - k_i\, a_i \text{ -}(g_i+p_i)\,\mu_i\, \exp\left(-\frac{a_i}{\mu_i}\right)\right]=$$

$$=600-7a_1-750\exp(-a_1/50)+$$
$$+1200-7a_2-160\exp(-a_2/80)$$

and

$$C(a_1,a_2) =$$

$$=19\left[80\exp(-a_2/80)+400/3\exp(-a_1/50-a_2/80)-\right.$$
$$\left.-640/3\exp(-a_1/80-a_2/80)\right]+$$
$$+12\left[50\exp(-a_1/50)-400/3\exp(-a_1/50-a_2/80)+\right.$$
$$\left.+250/3\exp(-a_1/50-a_2/50)\right] \quad (4.4)$$

For independent locations we get an optimal AD a°_i as the solution of the equation $\delta G^\circ/\delta a_i = 0$ or

$$F_i(a^\circ_i) = (g_i + p_i - k_i)/(g_i + p_i),\ i = 1, 2. \quad (4.5)$$

In the case of an exponential distribution function $F_i()$ with expectation μ_i it follows from equ. (4.5) that

$$a^\circ_i = -\mu_i\, \ln\left[k_i/(g_i+p_i)\right], \quad i=1,2. \quad (4.6)$$

Furthermore, from (4.3) and (4.6) it follows that

$$G^\circ(a^\circ_1, a^\circ_2) = \sum_{i=1}^{2}\left[g_i\,\mu_i - k_i\,(a^\circ_i + \mu_i)\right]. \quad (4.7)$$

For the Example 4.1 the equations (4.6) and (4.7) yield

$\mathbf{a}^0 = (38.107; 83.986)$ with an expected reward of $G^0(\mathbf{a}^o) = 35.349$ monetary units.

We notice that for location 1 we have an expected reward of -16.749 monetary units, i.e. for location 1 the independent work is a losing deal. The corresponding results for the 2-location model are

$$\mathbf{a}^* = (41.6; 90.2),$$

and

$$\mathbf{G(a^*)} = 209.006 \text{ monetary units.}$$

Comparing the results for the two models we can see that the co-operation leads to a profit increase of 173.657 monetary units. In other words, the co-operation of the two locations raises the initial profit to 591.26%. On the other hand, the optimal transportation capacity A* (and thus the optimal fleet size) for the co-operating locations is greater than $A^o = a^0_1 + a^0_2$ - the sum of optimal transportation capacities in the case of independent locations. Various numerical experiments have shown that these tendencies will be stronger with the increasing number N of locations.

We get further interesting results if we solve the 2-location model for different values of k_1 for instance (see Table 4.1):

1. For values of k_1 which violate condition (RA) from Section 3 in a sufficiently large size the whole transportation capacity is located in one location only. If $k_1 + c_{12} \leq k_2$, i.e. if $k_1 \leq 6$ then we can expect that for low values of k_1 all capacity will be concentrated in location l. From Table 4.1. we see that this is the case for $k_1 \leq 5$. On the contrary, if $k_2 + c_{21} \leq k_1$, i.e., if $k_1 \geq 10$ then we can expect that all capacity will be concentrated in location 2. Again, from Table 4.1 we see that this is true for $k_1 \geq 9$. Thus condition (RA) is only a necessary but not a sufficient condition for non-existence of degenerate allocation decisions.
2. The optimal capacity a^*_1 as well as the optimal expected reward are decreasing functions of the cost parameter k_1.
3. For $k_1 = 0$ it is optimal to have in location 1 an infinite transportation capacity, and in location 2 no capacity. The expected reward is equal to the expected profit $g_1\,\mu_1 + g_2\,\mu_2 = 1\,800$ minus the expected transfer cost $c_{12}\,\mu_2 = 80$, i.e., it is equal to 1720 (cp. Table 4.1).
4. The benefit from co-operation decreases with increasing k_1 whereas the percentile benefit increases with k_1.

Obviously the observed dependencies on k_1 hold also for k_2. However, we have to notice that the discussed dependencies are concluded from the data in Table 4.1, i.e. from data for a special example.

Table 4.1 - Results for different values of k_1.

k_1	a^*_1	a^*_2	$G(a^*)$	a^o_1	a^o_2	$G(a^o)$	$G(a^*)$-$G(a^o)$
10	0.00	121.48	186.78	20.27	83.99	-50.17	236.95
9	0.00	121.48	186.78	25.54	83.99	-27.77	214.55
8	5.89	117.28	187.51	31.43	83.99	0.66	186.85
7	41.60	90.20	209.00	38.11	83.99	35.35	173.65
6	131.96	16.87	289.53	45.81	83.99	77.21	212.32
5	167.35	0.00	445.60	54.93	83.99	127.44	318.16
4	188.04	0.00	622.96	66.09	83.99	187.75	435.21
3	213.93	0.00	823.39	80.47	83.99	260.68	562.71
2	249.38	0.00	1 053.93	100.75	83.99	350.61	703.32
1	308.25	0.00	1 329.51	135.40	83.99	466.70	862.81
0.5	365.75	0.00	1 496.41	170.06	83.99	542.07	954.34
0.1	496.80	0.00	1 662.26	250.53	83.99	622.05	1 040.21
0.01	682.10	0.00	1 712.38	365.66	83.99	647.94	1 064.44
0.001	866.60	0.00	1 719.05	480.79	83.99	651.16	1 067.89
0.00	∞	0.00	1 720.00	∞	83.99	652.10	1 067.90

5. CONCLUSION

In the present paper we have developed an approximate solution for the optimal allocation problem of transportation capacities in a multi-location system with redistribution. The proposed solution is an approximate one because the underlying model has discrete time, i.e. we have the planning horizon in the model divided into periods. One consequence of this time discretisation is that controlling actions are possible only at the beginning and at the end of a period. The usual approach for the continuous time case is the queuing model approach. However, for traffic networks we can hardly expect exact solutions. To solve an adequate problem, the network is decomposed into more or less independent nodes (see e.g. [DUHA97]: Thus we have again an approximate solution. Furthermore, in these models no redistribution between all locations is allowed. An interesting topic for future research is to verify and to compare the quality of both approaches by simulation. Another course for future research is to generalise the presented model in at least three directions. Usually not all information on the stochastic demand is known. Thus an adaptive control for the multi-location system makes sense (cp. [KÖCH88]). Secondly, since the criterion function G( ) from Section 3 is not given in an analytical form for more than two locations, we have to apply some approximation approaches (e.g. as in [KÖCH82]) or some searching methods as for instance Genetic Algorithms in [KÖAR96]. Thirdly, in reality we do not have linear cost functions. Often, we have non-linear functions or some fixed costs. For such cases the Genetic Algorithm approach seems to be the best one at the moment.

ZUSAMMENFASSUNG

EIN APROXIMATIVES MODEL ZUR FUHRPARK-BERECHNUNG UND REDISTRIBUTION

Beim analysierten Problem der Fuhrparkleitung handelt es sich um das Phenomen der Fahrzeug-alokation und um deren Neuverteilung. In dieser Arbeit werden Ergebnisse multi-lokativer-Modelle, appliziert an konkreten Problemen, erläutert:

1. *Vorteile der optimalen Distribution*
2. *Durchstudierte Konkavität der erwarteten Gewinne*
3. *Die Arbeit erläutert eine Beispiellösung bei unabhängigen kooperativen Subjekten in Form der Entscheidung mit Rücksicht auf den Standpunkt. Wegen der diskreten Berechnungen sind die Ergebnisse aproximativ.*

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