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Science in Traffic
Preliminary paper
U. D. C. 656.01:681.51(086.48)
Accepted: Mar. 30, 1999
Approved: Feb. 15, 2000

FLUID MODELS IN THE TRAFFIC FLOW THEORY

ABSTRACT

This paper presents a survey of results concerning continuum (fluid) models in the theory of traffic flow. We begin with the basic LWR model from 1955-56 and describe the benefits and deficiencies of that model. Afterwards we present some new models developed over the period from 1971 (Payne) until 1999 (Aw and Rascle) in attempt of correcting the deficiencies of classical LWR model

KEY WORDS

Traffic flow, fluid models, LWR model, second order models, shock waves.

1 INTRODUCTION

An objective of traffic engineering is to control the traffic flow on some network of roads to reduce the congestion and minimise the undesirable side effects (pollution) by redesigning traffic signs or network itself. To do so, we have to understand the nature of the traffic flow and be able to predict the response of the system to possible changes. The most accurate way to understand the reaction of the system is to perform simulations based on some, properly chosen, models. Choice of an appropriate model is the most important part of such study. It depends on the nature of the system and on particular effects that we want to analyse with our simulation.

Generally speaking, traffic flow models can be divided in microscopic, considering the motion of every individual vehicle, and macroscopic, considering the global (averaged) behaviour of a traffic flow. Microscopic approach, like the car-following model (see [24]), for large traffic systems does not seem to be feasible. Deterministic macroscopic models, called the fluid models, that we are about to present here, were initially developed for large highways (as in [14], [20]), where the average properties of drivers become visible. Later, fluid models have been adapted to heavy traffic networks with strong congestions, like in large cities (see e.g. [3], [4], [9], [10]). Continuum models can now be applied to almost any road traffic situation. The idea of such models is to treat the traffic flow as a flow of some granular fluid consisting of uniform particles (averaged vehicles) and to derive the equations of motion analogous to those from fluid mechanics. Such models are applicable to measure and analyse on a large scale such that a single vehicle is a negligibly small quantity. Such large-scale approximations are sometimes crude estimates and one is forced to use more refined stochastic models. However, only seldom does an engineer need to make the calculations of such high precision. Very often a decision as to whether a proposed traffic system will work well or create congestions can be seen from this deterministic approximations given by fluid models.

Another type of macroscopic models can be derived by stochastic approach, similar to the methods used to drive the Boltzmann equation in kinetic theory of gases. That approach, developed essentially by Prigorgine in the late fifties and early sixties, also leads to a macroscopic model described by integro-differential equation of Boltzmann type. The unknown function in their model is the velocity distribution function $f(x, v, t)$ defined such that $f(x, v, t)dx\,dv$ represents the probability of a car being in the small element of the road dx and moving with the velocity in range dv at time t. Such function includes information like performance characteristics of cars and drivers' wishes. However, Prigorgine's model, due to its complexity, is not as popular in the theory of traffic flow as the LWR model.

We restrict our presentation to fluid (continuum) macroscopic models.

2 LWR MODEL

2.1 Conservation law

In 1955 in their celebrated paper [14] Lighthill and Witham derived a simple hyperbolic conservation law for the traffic flow based on vehicle conservation principle. The same model was found few months later independently by Richards [20]. We present here only the simple LWR model used for describing the traffic flow on unidirectional roads with no junctions. An analysis of the two-way roads can be done in the same spirit (see [3], [22], [6]) but it leads to a system of conservation laws which makes its mathematical study rather complicated for this presentation. Based on the same ideas a model for network containing junctions can be derived, but it requires an additional entropy condition on each junction which, again, makes the mathematical study quite technical and complicated (see [10] for details). We begin by mathematical description of a simple LWR model. The continuum model for traffic flow does not allow individual tracking of cars, but it describes the dynamics of the macroscopic density of cars. Therefore, the unknown quantity in our problem is the car density. The fundamental assumption of all continuum models is that we can properly define the density function ρ. It should be noticed that the number of vehicles on some section of the road at some given time t is a discontinuous function, which is highly inconvenient for mathematical analysis. To overcome that difficulty we consider its smooth approximation $N(t)$ instead. The density of vehicles, defined as the number of vehicles per unit of length[1], is now related to $N(t)$ by simple integral with respect to x. More precisely, on some section of road with length L, from point x_0 to point $x_0 + L$, the number of vehicles at some time instant t, is given by

$$N(t)=\int_{x_0}^{x_0+L}\rho(x,t)dx \ . \qquad (1)$$

Our next task is to define the flow q. Quantity q is the number of cars by unit of time that pass the observer. Having defined the density ρ we can see that the traffic flow satisfies the relation

$$q(x,t)=\rho(x,t)\,v(x,t),$$

where v is the traffic speed.

The basic assumption of the LWR model is that

$$v=V(\rho) \ . \qquad (2)$$

It is clear that v and ρ are related because v decreases if the traffic is denser and increases if the density is lower. However, we could have some more general relationship $v=V\left(x,t,\rho,\frac{\partial\rho}{\partial x}\right)$. The assumption (2) simplifies the model and makes it easier to solve. The main deficiencies of that model are:

1. The speed at some point (x, t) depends only on the value of density computed at that same point, which means that a driver reacts only to the change of density around him and not further in front of him. Some authors argue that this is a reasonable assumption because, even though he sees the congestion in front of him, he does not react until he reaches the part of the highway where the congestion takes place. This assumption results with some discontinuous, abrupt changes of density (so called shocks). We shall return to this discussion in section 2.3.
2. The driver reacts immediately to the change of density, i.e. the driver's reaction time is zero and the reaction of the vehicle in instantaneous. In fact, we should take $v=V(\rho(x,t-T))$, where T is the reaction time. Again, one can argue that the continuum model is used for long time simulations and the reaction time T is negligible compared to the considered time interval.

To choose the convenient relationship between v and ρ we define v_{max} as the maximal speed on the considered highway. It is usually the quantity imposed by traffic signs and may vary on different sections of the road. For simplicity we assume that it is constant. We also define the maximal density ρ_{max}, also called the bumper to bumper density. If ρ takes the value ρ_{max} on some part of the highway, the vehicles are not able to move, i.e. $V(\rho_{max}) = 0$. Several different models have been proposed by taking different functions $V(\rho)$. The most popular ones are:

– Greensheld's model

$$V(\rho)=v_{max}\left(1-\frac{\rho}{\rho_{max}}\right) \qquad (3)$$

– Polynomial (or Pipes-Munjal's) model

$$V(\rho)=v_{max}\left[1-\left(\frac{\rho}{\rho_{max}}\right)^k\right] \qquad (4)$$

– Greenberg's model (v_g = empirical constant)

$$V(\rho)=v_g\ \ln\left(\frac{\rho_{max}}{\rho}\right) \qquad (5)$$

– Underwood's model

$$V(\rho)=v_{max}\ \exp\left(-\frac{\rho}{\rho_{max}}\right) \qquad (6)$$

– California model (v_c = empirical constant)

$$V(\rho)=v_c\left(\frac{1}{\rho}-\frac{1}{\rho_{max}}\right) \qquad (7)$$

To derive our differential equation we start from the car conservation principle. Let $N(t)$ be the number of cars on some section of the road $[x_1,x_2]$ at time t. To compute the number of cars at that same section of the road at time $t + \Delta t$ we need to deduce from $N(t)$ the number of cars that passed the point x_2, and that number, for small Δt, equals the flow $q(x, t_2)$ times Δt.

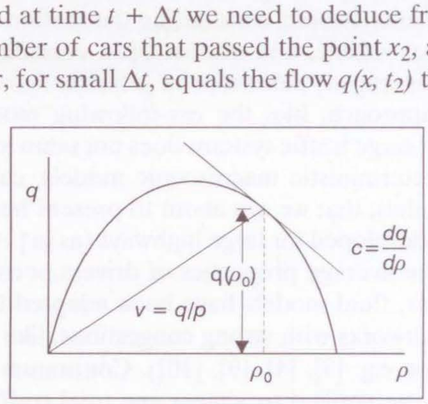

Figure 1 - Traffic flow curve

On the other hand, we need to add those cars that crossed the point x_1 and entered $[x_1, x_2]$, i.e. $q(x, t_1)\Delta t$. We obtain the relation

$$N(t+\Delta t)=N(t)+[q(x,t_1)-q(x,t_2)]\Delta t \quad (8)$$

implying (recall that N is smooth)

$$\frac{dN(t)}{dt}=q(x,t_2)-q(x,t_1) \ .$$

But, due to (1) we get

$$\frac{dN(t)}{dt}=\int_{x_1}^{x_2}\frac{\partial\rho}{\partial x}(x,t)\,dx$$

Since

$$q(x,t_1)-q(x,t_2)=\int_{x_1}^{x_2}\frac{\partial q}{\partial x}(x,t)\,dx$$

we arrive at

$$\int_{x_1}^{x_2}\left(\frac{\partial\rho}{\partial t}(x,t)+\frac{\partial q}{\partial x}(x,t)\right)dx=0$$

for arbitrary x_1, x_2. But then we necessarily have the local conservation law

$$\frac{\partial\rho}{\partial t}+\frac{\partial q}{\partial x}=0 \quad (9)$$

for any point (x, t). We note that in all the above models the function $\rho \mapsto q(\rho) = V(\rho)\rho$ is concave. In California model the function $q(\rho)$ is linear (and therefore concave) which makes the study of (9) trivial. In all the other models $q(\rho)$ is non-linear but strictly concave. For all those models, except Greenberg's model (5), q' is bounded. Greenberg's model is one of the earliest models, based on the data from the Lincoln tunnel in New York where the density is high. Therefore, its singular behaviour for low densities had no importance for that particular problem. On the other hand, that model is meaningless for problems with low densities unlike Underwood's model which fits very well experimental curves in case of low densities. It is commonly assumed in the traffic flow analysis that $q(0) = q(\rho_{max}) = 0$ and that q attains its maximum $q_{max} = q(\rho_c)$ (=the capacity of the road), for some $\rho_c \in \langle 0, \rho_{max}\rangle$. The usual empirical curve for flow versus density is similar to the one in Figure 1 (so called *fundamental diagram of road traffic)*. Such curve is obviously strictly concave. This places our law (9) in class of hyperbolic conservation laws. If we prescribe some initial density ρ_0 at time $t = 0$ we get the Cauchy's problem for hyperbolic partial differential equation (PDE) of first order

$$\frac{\partial\rho}{\partial t}+\frac{\partial q(\rho)}{\partial x}=0 \quad ,x\in\mathbf{R} \quad ,t>0 \quad (10)$$

$$\rho(0,x)=\rho_0(x) \quad ,x\in\mathbf{R} \quad (11)$$

By solving it we can simulate the evolution of the car density in time. We assume, in the sequel that q is strictly concave, smooth and that q' is uniformly bounded.

2.2 Method of characteristics

In some cases the problem (10)-(11) can be solved by method of characteristic. Characteristics for PDE are curves (in this case straight lines) in (x, t) plane on which the solution has constant values, i.e. curves $t \mapsto x(t)$ such that $\rho(x(t),t) = const.$ By deriving with respect to time t we obtain

$$\frac{d}{dt}\rho(x(t),t)=\frac{\partial\rho}{\partial t}(x(t),t)+\frac{\partial\rho}{\partial t}(x(t),t)\frac{dx(t)}{dt}=0 \ . \quad (12)$$

Comparing (12) with (10) we get the ordinary differential equation for $x(t)$ in the form

$$\frac{dx(t)}{dt}=q'\{\rho[x(t),t]\}=q'\{\rho_0(x(0))\} \quad (13)$$

leading to

$$x(t)=q'\{\rho_0(x(0))\}t+x(0) \ . \quad (14)$$

Along any characteristic the solution ρ retains the constant value $\rho(x, 0) = \rho_0(x)$. At each point $(x, 0)$ we draw a straight line with the characteristic slope $q'(\rho_0(x)$ and such procedure gives the graphical construction of the solution. For an arbitrary point (x, t) it suffices to find the characteristic

$$x-\xi=q'(\rho_0(\xi))t \quad (15)$$

and to pose $\rho(x, t) = \rho_0(\xi)$. We have reduced our problem (10)-(11) to the algebraic equation (15). However, the function $\rho_0(\xi)$ appearing in (15) (the initial density) can be quite complicated and (15) can be difficult or even impossible to solve.

Let us consider one more complicated example, where the characteristics do not immediately give the solution.

Example 1 *Consider the situation where there is a red traffic light at point* $x = 0$. *Then the density in front of the traffic light (i.e. for* $x < 0$) *is some* ρ_l *and the density behind the traffic light is equal to 0. The situation of interest is when* $\rho_l = \rho_{max}$ [2]. *At the initial time* $t = 0$ *the light turns to green and the vehicles start moving until the traffic light becomes red again. Such situation can be described by the initial density*

$$\rho_0(x)=\begin{cases}\rho_{max} & for\ x<0\\ 0 & for\ x>0\end{cases} \quad (16)$$

where ρ_{max} *the bumper-to-bumper density. Characteristics for such situation are drawn in Figure 3. It appears that there are no characteristics in sector G and the solution cannot be constructed by solving (15). In fact, one can construct characteristics in sector G but by a slightly different approach. Since all (possible) characteristics necessarily pass through the origin (i.e.* $x(0) = 0$), *an integration of (13) implies that those characteristics have the equation*

$$x=q'(\rho(x,t))t \quad (17)$$

leading to $\rho(x, t) = (q')^{-1}(x/t)$, *for any* $(x, t) \in G$. *We notice that* $q'' < 0$ *so that* q' *is always invertible. Now our* *problem (10)-(11) has a solution that can be written in the form*

$$\rho(x,t)=\begin{cases}\rho_{\max} & \text{for } x<q'(\rho_{\max})t\\ (q')^{-1}(x/t) & \text{for } q'(\rho_{\max})t<x<q'(0)t\\ 0 & \text{for } x>q'(0)t\end{cases}. \quad (18)$$

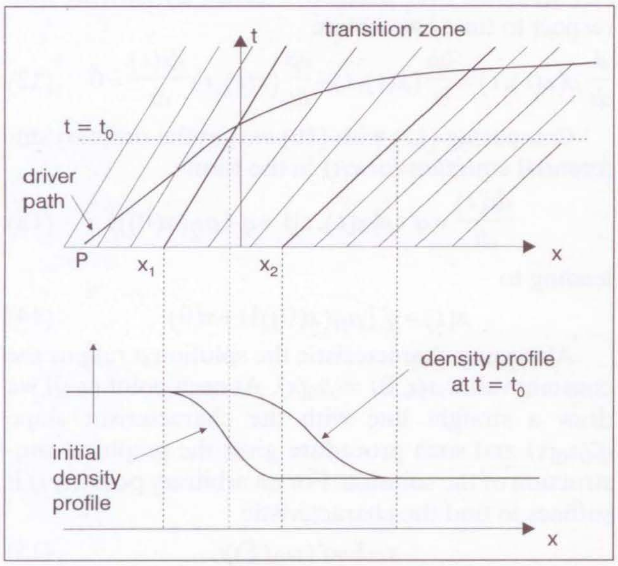

Figure 2 - Graphical construction of car density using Cauchy's characteristics

Such a solution is called the rarefaction (or expansion) wave and its characteristics are represented graphically in Figure 4. Let us compute the motion of the queue of cars in front of the traffic light. The velocity of the car with co-ordinates (x, t) is given by the velocity field

The car that was initially at position $-x_0$ (i.e. x_0 meters in front of the traffic light) stays still until the wave, propagating the information of the change of light, reaches it. After that time $t = -x_0/q'(\rho_{\max})$ the car starts moving at the velocity given in the rarefaction region G by

$$\frac{dx}{dt}=v(x,t)=V(\rho(x,t))=V\{[q']^{-1}(x/t)\} \ . \quad (19)$$

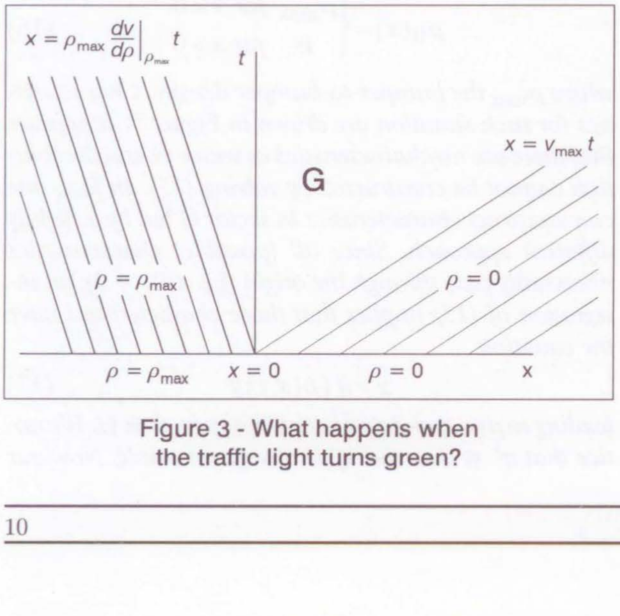

Figure 3 - What happens when the traffic light turns green?

To find the trajectory x(t) we need to solve (19) with the initial condition

$$x(-x_0/q'(\rho_{\max}))=-x_0 \quad (20)$$

In case of the simplest Greensheld's model (3) the solution equals

$$x(t)=v_{\max}t-2(x_0 v_{\max} t)^{1/2}$$

The car velocity is then given by

$$v=v_{\max}-(x_0 v_{\max}/t)^{1/2}$$

How long does it take the car to actually pass the light? It suffices to solve the equation x(t) = 0. For Greensheld's model the solution is $t = 4x_0/v_{\max}$ suggesting that it takes 4 times longer than if the car were able to move at the maximum speed. Consequently, if the light stays green until time T the number of cars that will pass the traffic light is $T\rho_{max}v_{max}/4$. For more complicated models the Cauchy problem (19)-(20) has to be solved numerically (using some standard ODE solver such as Runge-Kutta solvers in Maple and Mathematica packages). The car path can also be estimated by method of isoclines without actually solving (19).

Not only that (15) does not need to have the solution, as in example 1, but it can have several solutions, meaning that our construction gives several characteristics passing through the point *(x, t)*. In that case it is not clear which value to choose for $\rho(x, t)$, since on each of those characteristics ρ should have some different constant value. In fact, such a situation always happens if the initial density ρ_0 is decreasing.

Example 2 *Suppose that the initial density is given by*

$$\rho_0=\begin{cases}\rho_l & \text{for } x<0\\ \rho_r & \text{for } x>0\end{cases} \quad (21)$$

with $\rho_r \gg \rho_l$. In situation under consideration the initial traffic configuration consists of the light traffic zone

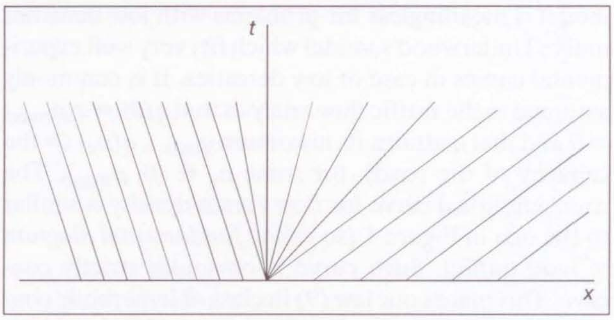

Figure 4 - Rarefaction wave

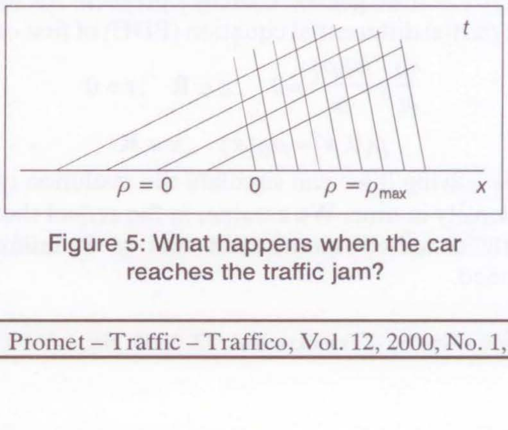

Figure 5: What happens when the car reaches the traffic jam?

x < 0 and the heavy traffic zone x > 0. The vehicles travel fast through the light traffic zone until they reach the heavy traffic zone (the bottleneck) where the average speed is lower and they have to brake. That causes perturbation of the density that travels like a shock wave in the direction opposite to the direction of the traffic flow. The front of our shock wave is, in fact the "interface" between the zone of light traffic and the zone of heavy traffic. We are interested in evolution of that wave in time. This effect is also called the wave phenomenon of automobile brake lights. Indeed, when the first car reaches the traffic jam the driver applies his brakes. That causes a chain reaction and the drivers behind him apply their brakes one-by-one. If you are driving behind the traffic jam, the brake light appears to travel in your direction, i.e. in the direction opposite to the traffic flow direction. From the point of view of an observer that is not moving with the traffic (i.e. in Eulerian system) the brake light wave can travel in the direction opposite to the traffic flow direction or in the traffic flow direction, depending on q(ρ), but from the point of view of one of the drivers (i.e. in Lagrangian system) its direction is always opposite to the traffic flow.

In such situation, characteristics drawn in Figure 5, intersect each other and our method does not give the solution. To solve that problem we need to dig deeper into the theory of hyperbolic conservation laws. We shall come back to this example in the following section.

2.3 Weak solutions and the entropy condition

Each continuously differentiable function ρ that satisfies the equation (10) and the initial condition (11) is called the classical solution for (10)-(11). Unfortunately, even if the initial density ρ_0 is smooth, Cauchy's problem (10)-(11) in general, does not admit a classical solution. In fact, situations like the one described in Figure 2, where the unique classical solution exists, are rare. In other words, situations like the ones in Figures 3 and 4, when the classical solutions do not exist are quite frequent. Therefore, we need to give the weaker definition of the solution. The basic idea is to multiply (10) by some smooth function ϕ, chosen such that $\phi = 0$ outside some bounded subset of *(x, t)* plane (i.e. compactly supported), and then to perform a simple partial integration. That way both derivatives in (10), $\frac{\partial}{\partial x}, \frac{\partial}{\partial t}$ pass to ϕ. In such a formulation of the problem, functions ρ and $q(\rho)$ appear with no derivatives so that they do not need to be differentiable. Those functions do not even need to be continuous and we only request that they are integrable on any bounded subset of *(x,t)* plane. More precisely:

Definition 1 *We say that a bounded measurable function is a* weak solution *for problem (10)-(11) if*

$$\int_0^{+\infty}\int_{-\infty}^{+\infty}\left(\rho(t,x)\frac{\partial\phi}{\partial t}(t,x)+q(\rho(t,x))\frac{\partial\phi}{\partial x}(t,x)\right)dtdx+ +\int_{-\infty}^{+\infty}\rho_0(x)\phi(0,x)dx=0\ , \tag{22}$$

for any compactly supported differentiable function ϕ.

Obviously, any classical solution is also a weak solution. Such solution still satisfies the conservation law in integral form (8). Weak solution always exists. In fact we have:

Theorem 1 ([11]) *If the initial density ρ_0 is bounded measurable function and the flow q is strictly concave function of ρ, then (10)-(11) has, at least one weak solution.*

The problem is that we can have several weak solutions. Indeed, it is easy to verify that the problem from example 1 has infinitely many weak solutions.

Example 1 *(continued) It is a simple exercise to verify that, for any $\rho_m \in \langle\rho_l, \rho_r\rangle$ each function of the form*

$$\rho(x,t)=\begin{cases}\rho_l & for\ x<s_m t\\ \rho_m & for\ s_m t<x<q'(\rho_m)t\\ (q')^{-1}(x/t) & for\ q'(\rho_m)t<x<q'(0)t\\ 0 & for\ x>q'(0)t\end{cases},$$

where and $s_m = \rho_l/2$, is a weak solution for problem (10)-(11) with initial condition (16). However, those solutions are not physically relevant and the only solution that corresponds to our real-world situation is (18).

The question is how to pick the physically relevant solution (18), i.e. the solution that describes our traffic flow.

It turns out that those difficulties are due to the fact that in LWR model the drivers react only to the local density. By assuming that, we have neglected the driver's anticipation and ability to react to density changes further ahead. If $\frac{\partial\rho}{\partial x}(x,t)>0$ the density is increasing in front of point *x* which should make a driver slow down. On the contrary, if $\frac{\partial\rho}{\partial x}(x,t)<0$ the density is decreasing and the driver should accelerate. It seems reasonable to modify LWR model by taking

$$q=V(\rho)\rho-\nu\frac{\partial\rho}{\partial x}\quad ,\nu>0 \tag{23}$$

where the small parameter ν measures the ability of the driver to react to those distant changes (we shall see in the last section that such modification can, in some carefully chosen situations, be contradictory). Such diffusive (or viscous, in analogy to the fluid mechanics) term makes our problem (10)-(11) parabolic and consequently easier to solve. In addition, it is smoothing our solution. This new problem, for smooth initial density ρ_0,

$$\frac{\partial \rho_\nu}{\partial t}+\frac{\partial q(\rho_\nu)}{\partial x}=\nu\frac{\partial^2\rho_\nu}{\partial x^2} \quad ,x\in\mathbf{R} \quad ,t>0 \quad (24)$$

$$\rho_\nu(0,x)=\rho_0(x) \quad ,x\in\mathbf{R} \ , \quad (25)$$

unlike our hyperbolic problem (10)-(11), has a unique classical solution ρ_ν. If ρ_0 is not smooth, but only bounded, the solution is not classical any more but weak. However, it is still unique. We expect that

$$\lim_{\nu\to 0}\rho_\nu=\rho$$

where ρ is some solution of "inviscid" problem (10)-(11). It turns out that this solution, obtained as the limit of the sequence of "viscous" solutions ρ_ν is the physically relevant solution that we are looking for. In analogy with gas dynamics such solution is called *the entropy solution.* The weak entropy solution has the same physical meaning as the classical solution, i.e. it satisfies our conservation law, only, not in the differential form (9) but in the integral form (8). There are some more practical methods to pick out the entropy solution. The following one is a good practical choice for scalar conservation laws:

Definition 2 *Let η be an arbitrary strictly concave function (entropy function) and let ψ be such that $\psi'(\rho)=\eta'(\rho)q'(\rho)$ (entropy flux). The function ρ is an entropy solution of Cauchy's problem (10)-(11) if*

$$\frac{\partial\eta(\rho)}{\partial t}+\frac{\partial\psi(\rho)}{\partial x}\geq 0$$

A simple example of an entropy pair would be

$$\eta(\rho)=-\frac{1}{2}\rho^2 \ , \ \psi(\rho)=-\int_0^\rho \xi q'(\xi)d\xi \ ,$$

Theorem 2 ([11]) *Under the conditions of theorem 1, the problem (10)-(11) has a* **unique** *entropy solution.*

In our two characteristic examples (example 1, example 2) the entropy solutions are easy to find.

Example 1 *(continued) As we have already mentioned for example 1 the rarefaction wave (18) is the (unique) entropy solution (see [1]). It is easy to verify that such solution is continuous.*

Example 2 *(continued) For the initial density (21) the entropy solution is easy to compute and it reads*

$$\rho(x,t)=\begin{cases}\rho_l & for\ x<st\\ \rho_r & for\ x>st\end{cases}$$

where

$$s=\frac{q(\rho_l)-q(\rho_r)}{\rho_l-\rho_r}=\frac{[q]}{[\rho]} \quad , \quad [\cdot]=jump \ . \quad (26)$$

The above solution has a discontinuity in line $x = st$. Such line is called *shock.* Shock is discontinuous perturbation of density that travels with speed s. Shock speed can be computed by formula (26) called the Rankine-Hugoniot's jump condition. Whenever there is some abrupt perturbation on the car density, due to the traffic light change, some accident, traffic jam or any other reason, shock wave will appear. That formula is important for practical simulations because it enables computing the speed of propagation of such shock wave. One should not confuse the traffic speed v and the shock speed s. Not only that they are not equal but the shock can propagate in direction opposite to the direction of traffic.

Example 2 *(continued) For example, in Fig. 3, we have a situation when a vehicle moving through light traffic (with density $\rho_r \gg \rho_l$). The driver is forced to brake and so are all the drivers behind him. The point x_b where cars begin to brake is not fixed. We see that $x_b(t)$ is moving with the speed s. In fact $x_b(t) = st + x_b(0)$ is the shock wave with speed $v_E = dx_b/dt = s$ in Eulerian system, or $v_L = s - V(\rho_l)$ in Lagrangian system moving with the uniform traffic flow with velocity $V(\rho_l)$. Depending on the sign of $q(\rho_r) - q(\rho_l)$ the shock velocity s can be positive or negative i.e. the brake light wave can travel in direction of the flow or against it. If $\rho_r = \rho_{max}$ (i.e. the traffic has completely collapsed at some point due to some accident, an enormous crowd or simply red light, which are typical shock creating situations) then*

$$s=-q(\rho_l)/(\rho_{\max}-\rho_l)<0 \ ,$$

implying that

$$x_b(t)=-q(\rho_l)t/(\rho_{\max}-\rho_l)+x_b(0)$$

and the shock travels against the traffic flow. On the other hand, $v_L=\rho_r(V(\rho_r)-V(\rho_l))/(\rho_r-\rho_l)<0$ as we noted in the first part of example 2. At time t shock is at position $x_b(t)$. If some car is at distance d from the shock's origin $x_b(0)$ at time $t = 0$ we can compute the time when it will be caught by the shock $t = -d/v_L$. The shock can be visualised as the travelling taillight. We note that, in this case, we have $\rho_l < \rho_r$.

It turns out that the entropy solution can be recognised as the only solution that satisfies

$$\lim_{(x,t)\to shock-}\rho\leq\lim_{(x,t)\to shock+}\rho$$

on any shock. Ansorge [1], proposed to call this effect *the driver's impulse,* since it can be seen as the driver's wish to smooth up a discontinuous traffic density.

To end this section we notice that we have presented two examples where the solution can be computed analytically. In general, that cannot be done, and we need to apply some numerical methods. In fact, that is the case whenever the initial density ρ_0 *is increasing and it is not peacewise constant* (as it was in example 2). Numerical methods for hyperbolic conservation laws have been extensively studied over the last four decades, in particular for applications in fluid mechanics. However, there are only few papers on applications of numerical methods in theory of traffic flow (see. e.g. [15], [16], [12] and the references therein). Since rarefaction and compression waves are typical for the traffic flow, the non-uniqueness of the weak solution appears and we need to avoid the numerical schemes that converge to non-physical (entropy violating) solutions. The most popular methods for one-dimensional scalar conservation law are finite difference methods. To get the physically relevant solution we need to choose those methods that guarantee entropy solution, such as monotone or TVD schemes (see[11] or [13] for the precise definition). A reasonable choice would be, for instance, Lax-Friedrichs and Engquist-Osher's scheme (under the Courant-Friedrichs-Levy's condition) or the Godunov's method in its variants (like Roe-Murman's or Glimm's scheme).

3 SECOND ORDER MODELS

The basic assumption in LWR model was that the velocity v and the car density ρ are related by some given function V. In fluid mechanics the relation between v and ρ is not prescribed but we have the equation of mass conservation that reassembles our LWR model and an additional equation representing the conservation of momentum. Supposing that there is a relation between the pressure and the density (isentropic fluids) and that the flow is unidirectional, it gives a system of two equations with two unknowns, v and ρ. In analogy with fluid mechanics, the idea of second order models[3] is to try to mimic the momentum equation instead of prescribing the relation between v and ρ. It should be noticed that the conservation of momentum has no sense in theory of traffic flow. However, adding this second equation, derived by combining some microscopic models with LWR model, is an attempt to repair the basic deficiencies (1 and 2) of the LWR model.

The most popular second order model is **Payne's model** from 1971 derived in [18].

Starting from the car following equation

$$\frac{dx_{n+1}(t+T)}{dt}=V\left(\frac{1}{x_n-x_{n+1}}\right),$$

where T is the reaction time, V is as in the LWR model and x_k is the path of the k-th vehicle, and using the Taylor's expansion of both sides, he found the "momentum equation" in the form

$$\frac{\partial v}{\partial t}+v\frac{\partial v}{\partial x}=\frac{1}{T}[V(\rho)-v]-\kappa\frac{\partial \rho}{\partial x} \qquad (27)$$

where

$$\kappa=-V'(\rho)/2T\ \rho\geq 0.$$

System (10)-(27) is a 2 × 2 hyperbolic system. Prescribing the initial density and the initial velocity

$$\rho(x,0)=\rho_0(x)\quad,\quad v(x,0)=v_0(x)$$

under certain technical conditions on κ, ρ_0 and v_0, such system has a unique entropy solution. However, such solution cannot be constructed analytically for any reasonable initial data, as we were able to do in examples 1 and 2 for the LWR model. Its theory and numerical resolving are much more complicated. Even the definition of an entropy pair η, ψ requires some serious analysis.

Numerical applications of Payne's model reported mixed results (see [23] and [12]). According to [12] simulations based on Payne's model give results that match the experimental data better than (or at least as well as) those using the LWR model, but the small negative velocity occasionally appears when the density is close to the jam density.

C. Daganzo proved in [5] that one can always construct the initial (peacewise constant) data, for Payne's model, that lead to the negative, non-physical, car velocities. The reason for this unacceptable drawback seems to be that the anticipation factor involves the derivative of ρ only with respect to x. That way if, for instance, in front of the driver (travelling at speed v) the density increases with respect to x but decreases with respect to $x - vt$, Payne's model predicts that the driver would slow down. On the contrary, any reasonable driver would accelerate because, although the traffic is denser in front of him, it is travelling faster than him.

Several attempts have been made to resolve these inconsistencies. In [23] H. M. Zhang started from the micro-macro equation

$$\frac{dx(t+T)}{dt}=V(\rho(x(t)+\Delta,t)),$$

where $[x, x+\Delta]$ is the driver's reaction zone. Using Taylor's expansions, like Payne, and assuming that

$$\frac{\Delta}{T}=-\rho V'(\rho),$$

he arrived to equation (27) with $\kappa=\rho V'(\rho)^2$. According to [23] such model does not create the wrong-way travel like Payne's model.

Another approach to correct (27) was made by A. Aw and M. Rascle [2]. Neglecting the relaxation term $[V(\rho)-v]/T$ they proposed to replace $\partial p/\partial x$ by the convective derivative $\partial p/\partial t+v\,\partial p/\partial x$. Such modification eliminates completely the wrong-way travel effect.

SAŽETAK

FLUIDNI MODELI U TEORIJI PROMETNOG TOKA

U ovom radu dan je prikaz rezultata o kontinuiranim (fluidnim) modelima u teoriji prometnog toka. Počinjemo s osnovnim LWR modelom iz 1955.-56. i opisujemo sve njegove dobre i loše strane. Nakon toga opisujemo neke novije modele nastale u periodu od 1971. (Payne) do 1999. (Aw i Rascle) s ciljem ispravljanja nedostataka klasičnog LWR modela.

NOTES

1. see [4] for precise definition
2. if there is no long queue in front of the traffic light, people are not likely to worry about it

3. Term "second order" is mathematically incorrect since most of those models are first order PDE. Actually, from the mathematical point of view it would be more correct to call them "two equations models" since those are systems of two differential equations with two unknown functions. Nevertheless, we shall call those models " second order models", since it is accustomed in the engineering literature.

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