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VEHICLE ROUTING PROBLEM MODELS

ABSTRACT

The Vehicle Routing Problem cannot always be solved exactly, so that in actual application this problem is solved heuristically. The work describes the concept of several concrete VRP models with simplified initial conditions (all vehicles are of equal capacity and start from a single warehouse), suitable to solve problems in cases with up to 50 users.

KEY WORDS

Capacitated Vehicle Routing Problem, linear model, graph, transport

1. INTRODUCTION

The Capacitated Vehicle Routing Problem (CVRP) is the basic modification of the initial VRP problem. The VRP itself represents the determination of the optimal tour used by a group of vehicles in serving a set of users. In this problem all the users and their requirements are known in advance, the vehicles are identical and the common starting point is at the central warehouse. The only existing constraint is the capacity of individual vehicles. The function of objective is to minimize the total cost (e. g. the weight function of the number of routes and their total length or time) in serving all users.

Practice has shown that the application of the VRP model results in savings of 5 – 20% of the overall transportation costs. The application of the VRP model provides comprehensive application in traffic: delivery and collection of regular and special cargoes, public traffic, communal and catering services, transportation of disabled persons, dynamic direction of traffic, optimal regime of temporary traffic regulation, etc.

2. CAPACITATED VEHICLE ROUTING PROBLEM (CVRP)

CVRP can be described as a problem in the theory of graphs which states as follows:

let $G=(V, A)$ be a complete graph, $V=\{0,..., n\}$ a set of nodes and A a set of links. Nodes $i=1, ..., n$ represent the users. Node with index 0 represents the warehouse. The non-negative price c_{ij} is related to the link $(i, j) \in A$ and represents the transportation cost between the users i and j. For CVRP three approaches to the problem modelling are known: vehicle flow model, cargo flow model and SP model.

The commonest way of modelling the vehicle routing problem is the usage of integer variables assigned to each line of the graph. The variables count how many times a vehicle has used a link. Such approaches have been known as the vehicle flow models. These are the commonest models, especially practical in cases when the price of solution can be simply combined as a sum of link prices or when the most important constraints can be defined by the members of the set of links and the related prices. The weakness of such models becomes obvious when the price of the solution is related to the overall sequence of the users on the route or the type of vehicle assigned to the route. The vehicle flow models are not suitable for the relaxation process of linear programs either, when the additional constraints are very narrow.

Another family of models is based on the flow of cargo. In such models new integer variables are added and they are related to the graph lines and represent the flow of cargo along the routes travelled by the vehicle. Only recently, the increase in the computer processing power has made these models interesting as the basis for exact calculation of optimal solution for CVRP.

The SP model generates a single binary variable for every feasible route, and then provides transition to the definition of the problem as an **SPP** problem *(Set Partitioning Problem)*. The main advantage of these models is that they enable very generalised setting of the costs of single routes, so that the costs can depend on the whole sequence of links and types of vehicles in various ways. The main problem is the exponential increase of the number of binary variables.

2.1. Vehicle flow models

The two-index model of vehicle flow uses ($O(n^2)$) binary variables x which mark whether a vehicle used a link in the optimal solution or not. The variable x_{ij} is assigned the value *1* if the link $(i, j) \in A$ (A is a set of links) belongs to the optimal solution, and if the link does not belong to the optimal solution x_{ij} is assigned the value *0*.

$$\min \sum_{i \in V} \sum_{j \in V} c_{ij} x_{ij} \quad \text{(1.1)}$$

$$\sum_{i \in V} x_{ij} = 1 \qquad \forall j \in V \setminus \{0\} \quad \text{(1.2)}$$

$$\sum_{j \in V} x_{ij} = 1 \qquad \forall i \in V \setminus \{0\} \quad \text{(1.3)}$$

$$\sum_{i \in V} x_{i0} = K \quad \text{(1.4)}$$

$$\sum_{j \in V} x_{0j} = K \quad \text{(1.5)}$$

$$\sum_{i \notin S} \sum_{j \in V} x_{ij} \geq r(S) \qquad \forall S \subseteq V \setminus \{0\}, S \neq \varnothing \quad \text{(1.6)}$$

$$x_{ij} \in \{0,1\} \qquad \forall i, j \in V \quad \text{(1.7)}$$

The constraints **(1.2)** and **(1.3)** indicate that precisely one vehicle arrives to the user and that one vehicle departs from the user. In accordance with this, the constraints **(1.4)** and **(1.5)** indicate that K vehicles arrive to and depart from the warehouse. The sufficient number of constraints is $2|V|-1$ since the last constraint is implicitly determined by the others. The so-called capacity cut constraints **(1.6)** in further text **CCC,** determine the connections and the capacity of the solution. CCC establishes that every cut $(V \setminus S, S)$ defined by the set of users S intersects the links and that the number of cut links is always greater than or equals $r(S)$, i. e. the minimal number of vehicles required to serve the set S **(Fig. 1)**.

The case in which $|S| = 1$ or $S = V \setminus \{0\}$. for **(1.6)** is considered, i. e. CCC which is then the weakened form of constraints **(1.2)**, **(1.3)**, **(1.4)** and **(1.5)**. In case of the cut it may be concluded that:

$$\sum_{i \notin S} \sum_{j \in S} x_{ij} = \sum_{i \in S} \sum_{j \notin S} x_{ij} \qquad \forall S \subseteq V \setminus \{0\}, S \neq \varnothing \quad \text{(1.8)}$$

The cut intersects the links in both directions equal number of times so that:

$$\sum_{i \notin S} \sum_{j \in S} x_{ij} \geq r(V \setminus S) \qquad \forall S \subset V \setminus \{0\}, 0 \in S \quad \text{(1.9)}$$

Alternative formulation leads to the constraint of the generalised subtour elimination - **GSEC** *(Generalized Subtour Elimination Constraints)* – **Figure 2.**

$$\sum_{i \in S} \sum_{j \notin S} x_{ij} \leq |S| - r(S) \qquad \forall S \subset V \setminus \{0\}, S \neq \varnothing \quad \text{(1.10)}$$

Constraints **(1.6)** and **(1.10)** have cardinality that increases exponentially with the number of users n. Due to such increase it is practically impossible to solve the vehicle routing problem in the general case by direct relaxation of a linear program. It is potentially possible to consider a smaller subset of constraints and to gradually add those constraints that are necessary, using one of the separation procedures. The considered constraints can be relaxed in the Lagrangian way as in (Fisher 1994) and (Miller 1995) or they can be directly solved by relaxation of a linear program as in «branch and cut» approach.

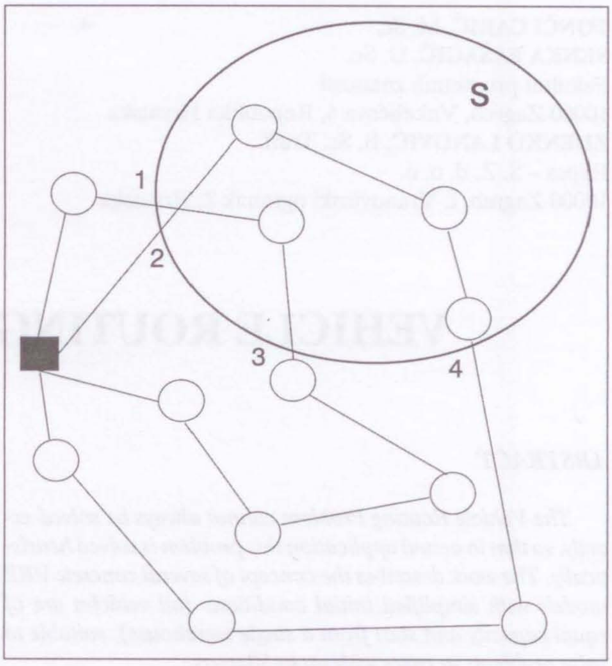

Figure 1 - Capacity cut

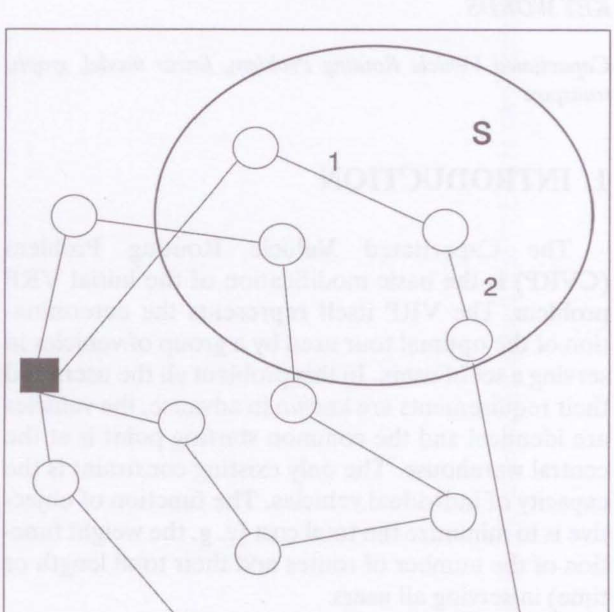

Figure 2 - Constraint of the generalised subtour elimination

2.2. Cargo flow models

The cargo flow model was first introduced in the work (Gavrin et al. 1957) for the oil distribution problem, and was later extended in the works (Gavish et al. 1979), (Gavish et al. 1982) on two versions for TSP and VRP. In this model, apart from the variables necessary for the two-index model of vehicle flows, a new set of continuous variables connected to each link is added. The model requires an expanded graph $G'=(V', A')$ created from graph G by adding the user $n+1$ which is another copy of the warehouse. The routes are represented by paths of connected links from user 0 to user $n+1$. Two non-negative variables of flows y_{ij} and y_{ji} are connected with link $(i, j)\in A'$. If the vehicle travels from i to j then y_{ij} represents cargo, and the variable y_{ji} the unused capacity of the vehicle along the link (i, j). Obviously, the relation of these two variables is $y_{ji} = C - y_{ij}$. The roles of variables change if the vehicle travels from j to i. Therefore, the equation $y_{ji} = C - y_{ij}$ is satisfied for every $(i, j)\in A'$.

For each route that represents a part of the feasible solution, the flow variables define two directed routes. One route represents one direction with variables representing cargo, whereas in the other direction these same variables represent the unused capacity. Thus, a vehicle can be assumed to start from the user 0 to the user $n+1$, departing from the user 0 with enough goods so as to deliver the required quantity to every user. The vehicle arrives empty to the user $n+1$. Let us assume another vehicle starting empty from the user $n+1$ and collecting goods from every user, equal in quantity to the user's requirement. **Figure 3** presents an example with 3 users and the vehicle capacity $C=20$, and the users' requirements are represented by the number next to each user i, j and k.

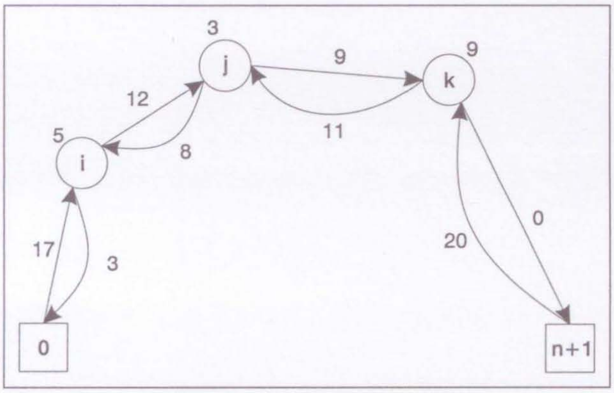

Figure 3 - Example of cargo flow model

Like in the two-index model of vehicle flow, the binary variables x are used, which mark whether the vehicle used the link in the optimal solution or not. The variable x_{ij} is assigned the value 1 if the link $(i, j)\in A'$ belongs to the optimal solution, and if the link does not belong to the optimal solution x_{ij} is assigned the value 0. The model looks as follows:

$$\min \sum_{i,j\in A'} c_{ij} x_{ij} \tag{2.1}$$

$$\sum_{i\in V'} (y_{ji} - yij) = 2d_i \tag{2.2}$$

$$\sum_{j\in V'\setminus\{0,n+1\}} y_{0j} = d(V'\setminus\{0,n+1\}) \tag{2.3}$$

$$\sum_{j\in V'\setminus\{0,n+1\}} y_{j0} = KC - d(V'\setminus\{0,n+1\}) \tag{2.4}$$

$$\sum_{j\in V'\setminus\{0,n+1\}} y_{n+1,j} = KC \tag{2.5}$$

$$y_{ij} + y_{ji} = Cx_{ij} \qquad (i,j)\in A' \tag{2.6}$$

$$\sum_{i\in V'} (x_{ij} + x_{ji}) = 2 \qquad \forall i \in V'\setminus\{0,n+1\} \tag{2.7}$$

$$y_{ij} \geq 0 \qquad (i,j)\in A' \tag{2.8}$$

$$x_{ij} \in \{0,1\} \qquad (i,j)\in A' \tag{2.9}$$

The constraint **(2.2)** determines that the difference between the sum of cargo flow variables related to arrival and departure of every user i equals the double value of the requirement of every user. The constraints **(2.3)**, **(2.4)** and **(2.5)** insure correct values of cargo flow variables towards the warehouse. The relation among the cargo flow variables is given by the relation **(2.6)**. The number of visits to every user is limited by the relation **(2.7)**.

2.3. SP model

The Set Partitioning Model – **SP**, was first proposed in the work (Balinski et al. 1964) and uses potentially the exponential number of binary variables. Each variable is connected with one of the feasible routes from the set of all routes $\Psi=\{H_1, \dots H_q, \}$ where $q=|\Psi|$. Every route H_j has its own cost c_j. The binary coefficient a_{ij} is assigned the value 1 if the user is visited, or 0 if the user is not visited. The binary variable x_j $j=1, \dots, q$ equals 1 if and only if the route H_j is part of the optimal solution. The model looks like this:

$$\min \sum_{j=1}^{q} c_j x_j \tag{3.1}$$

$$\sum_{j=1}^{q} a_{ij} x_j = 1 \qquad \forall i \in V'\setminus\{0\} \tag{3.2}$$

$$\sum_{j=1}^{q} x_j = K \tag{3.3}$$

$$x_j \in \{0,1\} \qquad \forall j = 1,\dots,q \tag{3.4}$$

The constraint **(3.2)** requires that every user is visited only once in the optimal solution and that the solution has K routes of constraint **(3.3)**. This is a very generalised model that can be very easily extended by new constraints.

The main drawback of the model is the explicit generation of all the possible routes, which is very impractical, and the "column generation" approach is a good signpost towards the solution in order to solve the linear program set in this way.

3. CONCLUSION

With today's best exact algorithms it is possible to solve the vehicle routing problem only in cases with fewer than 50 users, in such a way that the optimum in the biggest problems can be reached only in individual cases. The actual problems that need to be solved greatly exceed this number of users, so that the exact approach should be replaced by the heuristic approach.

The work presents the exact models with certain simplifications (vehicles of equal capacity starting from a single warehouse) to solve practical transportation problems. The efficiency of even these simpler models is confirmed by the fact that the application of the VRP models results in the reduction of the existing overall transportation costs by 5 – 20%.

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SAŽETAK

MODELI PROBLEMA USMJERAVANJA VOZILA

Problem usmjeravanja vozila nije uvijek moguće riješiti egzaktno pa se u realnim primjenama taj problem rješava heuristički. U radu je opisana koncepcija nekoliko konkretnih VRP modela s pojednostavljenim početnim uvjetima (sva vozila su istog kapaciteta i polaze iz jednog skladišta), pogodnih za rješavanje problema u slučajevima s manje od 50 korisnika.

KLJUČNE RIJEČI

problem usmjeravanja vozila s ograničenjima kapaciteta, linearni model, dijagram, prijevoz

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