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A MIXED INTEGER LINEAR PROGRAM FOR THE SINGLE DESTINATION SYSTEM OPTIMUM DYNAMIC TRAFFIC ASSIGNMENT PROBLEM WITH PHYSICAL QUEUE

ABSTRACT

In order to solve the system optimum dynamic traffic assignment problem, the whole link model with physical queue is used to formulate the single destination system optimum dynamic traffic assignment problem as a mixed linear program. A relationship between the cumulative curves and the wave speed presented by Newell (1993) is used to present a dynamic network model in considering spillback queue. And nonlinear constrains are relaxed into mixed linear constrains; the linear program software is used to solve the system optimum dynamic traffic assignment problem. A numerical example illustrates the simplicity and applicability of the proposed approach.

KEYWORDS

Dynamic System Optimum Traffic Assignment; physical queue; mixed integer linear programming

1. INTRODUCTION

With the development of Intelligent Transportation Systems, the dynamic traffic assignment problem has been one of the most important areas in transport research. Contrasting with its user equilibrium counterpart, the system optimal dynamic traffic assignment problems (SO-DTA) remain an underdeveloped important area. Such assignment can be divided into two classes: one class are the Path-based SO-DTA models; however, the research along this line is rather limited. One major reason is that solving of path-based SO-DTA models usually requires the calculation of the dynamic marginal path travel costs (PMC). However, the path cost mapping usually does not have an explicit functional form; the evaluation of PMC is not straightforward. Shen, W. (2007) gave a dynamic PMC function with point queue in a simple network without any divergence [1]; Chow, A. (2008) provided a detailed mathematical analysis and discussion of dynamic system optimal assignment when the deterministic queuing model is adopted [2]. Another class are the Link-based SO-DTA models. Wie, B. (1998) presented a non-convex control model of dynamic system-optimal traffic assignment [3]. Ziliaskopoulos, A.K. (2000) used the cell transmission model to formulate the single destination SO-DTA problem as a Linear Program [4]. Shen, W. (2008) reviewed the link-based SO-DTA formulations based on different traffic flow models and explored the impact of modeling details in the traffic flow model on the optimal solution of the SO-DTA model [5]. However, previous researches did not consider the link-based dynamic traffic flow models with the physical queue. In this study, the link-based SO-DTA formulations with physical queue are formulated as a mixed integer linear program.

The rest of the paper is organized as follows; firstly, a relationship between the cumulative curves and the wave speed is used to present a dynamic network model with physical queue in single destination network. Then, non-linear constrains are relaxed into mixed linear constrains, the linear program software is used to solve the SO-DTA. Finally, the numerical example in a network is given.

2. MODEL FORMULATIONS

Notation:

w_a^f, w_a^b = the forward and backward wave speed on link a.

l_a = link a length. k_a = congested density on link a.

S_a^{in}, S_a^{out} = entry and exit capacity on link a.

$C_a^{out}(k)$ = exit capacity on link a during interval k.

$X_a(k), Y_a(k)$ = possible inflow and departure flow rate on link a during interval k.

$u_a(k), v_a(k)$ = actual inflow and exit flow rate on link a during interval k.

$U_a(k), V_a(k)$ = cumulative arrivals and departures at link a until interval k.

$q_a(k)$ = queue vehicle number on link a until interval k.

$t_a(k)$ = travel time on link a for travelers entering this link at interval k.

We consider a road network G(N,A) composed of a finite set of nodes, N (an ordinary node set N_o, a merge node set N_m , and a diverge node set N_d), and a finite set of directed links, A (a set L_r of regular links and a set L_d of dummy links). Let a be a link, each dummy link $a \in L_d$ connects one origin to the network and has infinite capacity and zero free flow travel time. Let R be the set of origin nodes and let r represent an origin $r \in R$. Let s represent a single destination.

The studied horizon is discretized into m intervals of length δ such that $T = m \cdot \delta$. In other words, we have intervals, index k representing interval $[(k-1)\cdot\delta, k\cdot\delta)$. Here, it is assumed the study horizon is long enough to ensure all travelers can exit from the network after time T. On the other hand, it is also assumed that the value of δ is small enough so that the discrete-time model can approximate its continuous time counterpart. Further it is assumed that the flow rate, either specified by link or by path, is constant during a given interval.

2.1 Link flow propagation

Newell (1993) analyzed a relationship between the cumulative curves and the wave speed [6]. The flow-density relationship is described by only two wave speeds. The forward wave speed is similar to free flow speed under the point queue conditions. If the possible link departure flow exceeds the maximum capacity of link a, a queue is formed and the backward wave starts moving. The backward wave speed reflects the increasing queue length. Under the spillback queue concept, the link capacity may change over time since a queue physically backs up and may reduce the link capacity of the upstream link.

The following section shows the link flow propagation and node flow assignment under the spillback queue concept.

2.1.1 Possible link inflow and departure flow

Following M. Kuwahara (2001) and Guido Gentile (2005) [7, 8] the possible link inflow rate can be expressed as follows:

$$X_a(k) = \begin{cases} v_a\left(k - \dfrac{l_a}{w_a^b}\right) & \text{if } \mathrm{U}_a(k) > V_a\left(k - \dfrac{l_a}{w_a^b}\right) + k_a \cdot l_a \\ S_a^{in} & \text{otherwise} \end{cases} \quad (1)$$

where S_a^{in} is the entry capacity (link storage capacity) of link a, $k_a \cdot l_a$ is the link maximum storage space on link a. If a queue reaches the link entry at time interval k, the possible link inflow is restricted to the actual exit flow rate at time interval $k - l_a/w_a^b$, otherwise, $X_a(k)$ is equal to the link entry capacity.

The possible link departure flow rate at time interval k consists of two parts: one is the actual link inflow rate at time interval $k - l_a/w_a^f$, the other is the queue vehicle at time interval $k-1$ (in previous studies, the link queue is neglected). The equation is written as

$$Y_a(k) = \min\left\{S_a^{out}, u_a\left(k - \frac{l_a}{w_a^f}\right) + \frac{q_a(k-1)}{\delta}\right\}, \forall a, k \quad (2)$$

The queue vehicle number at time interval k can be expressed as follows.

$$\begin{aligned} q_a(k) &= \max\{0, \delta \cdot (Y_a(k) - v_a(k))\} = \\ &= \max\{0, q_a(k-1) + \delta \cdot (u_a(k - l_a/w_a^f) - v_a(k))\} \end{aligned} \quad (3)$$

The calculation method of the queue length is similar under the spillback queue and the point queue concepts; however, both meanings are distinctive in essence. Under the point queue concept, we assume a vehicle as a point without length; therefore a point queue is vertically formed at the exit point of link. However, a spillback queue is formed horizontally, and we cannot use the queue vehicle number to calculate explicitly the queue length, since the speed and density of the queuing vehicles vary over time and space as a function of exit capacity.

2.2 Node flow assignment

The node flow assignment model is used to determine the exit capacities of link, on the basis of the possible link inflow rate of the downstream links and of the turning flows. To simplify the exposition, we first consider only two typologies of nodes: "merging" and "diverging".

Considering the merging node n with entering links $a_1, \ldots, a_i$ and exiting link b. The actual exit flow during time interval k is given by

$$\begin{aligned} v_{a_i}(k) &= \min\{Y_{a_i}(k), \lambda_{b,a_i}(k) \cdot X_b(k)\} = \\ &= \min\{S_{a_i}^{out}, u_{a_i}(k - l_{a_i}/w_{a_i}^f) + q_{a_i}(k-1)/\delta, \\ &\quad ,\lambda_{b,a_i}(k) \cdot X_b(k)\} \\ &\forall a_i \in \prod(n), b \in \Psi(n), k, n \in N_0 \cup N_m \end{aligned} \quad (4)$$

where $\lambda_{b,a_i}(k)$ expresses the supply flow on link a_i from the downstream link b. We assume

$$\lambda_{b,a_i}(k) = S_{a_i}^{out} / \sum_n S_{a_n}^{out},$$

the $\lambda_{b,a_i}(k)$ is calculated using the method presented by M. Kuwahara (2001) and Guido Gentile (2005), too [7, 8]. Eq.(4) showing the actual exit flow during time interval k on link a_i may be subject to the restriction of the link departure flow as well as the possible inflow of the downstream link. When the upstream link number of node i is 1, then $\lambda_{b,a_i}(k) = 1$, since eq.(4) can further model the actual link exit flow at the ordinary node. $\Pi(n), \Psi(n)$ represent the upstream and downstream link set of node n, respectively.

Considering a diversion node n with entering link a and exiting links $b_1, \ldots, b_j$. The link exit flow is given by

$$\max\Big(\sum_{b_j} v_{a,b_j}(k)\Big) \tag{5}$$

$$\sum_{b_j} v_{a,b_j}(k) \le Y_a(k), \forall a \in \Pi(n), k, n \in N_d \tag{6}$$

$$v_{a,b_j}(k) \le X_{b_j}(k), \forall b_j \in \Psi(n), k, n \in N_d \tag{7}$$

Where, $v_{a,b_j}(k)$ represents the link turning flow from link a to link b_j during time interval k. Eqs.(5-7) use the program method in solving the maximum turning flow at diversion node n. Eq.(6) represents the total turning flow that cannot exceed the possible departure flow of the upstream link a. Eq.(7) representing the turning flow must be smaller than the possible inflow of the downstream link. Here, the link actual exit flow can be represented as follows:

$$v_a(k) = \sum_{b_j} v_{a,b_j}(k), \forall a, k \tag{8}$$

2.3 Origin link

At the network boundary, the travel demand generated at the origins during each time interval enters the dummy link [5]. The boundary constraint, namely, is as follows:

$$u_a(k) = q_r(k), \forall a \in L_d, r \in R, k \tag{9}$$

where $q_r(k)$ represents the traffic demand from origin r entering the network during time interval k.

2.4 Objective function

The total system travel cost is used for the objective function in the previous studies. However, the major difficulty in solving these SO-DTA formulations are non-linear. In this study, in order to model Linearity, the minimum of the total queue number is used in solving the traffic flow. The objective function can be expressed as follows:

$$\min\Big(\sum_a \sum_k q_a(k)\Big) \tag{10}$$

Constraints: (1–9)

3. MODEL LINEARIZATION

The major challenge in solving the above link-based SO-DTA formulations with physical queue are the non-convexity associated with the non-linear inequality constraints in the formulations. To bypass the non-convexity, these non-linearity inequalities are often relaxed into the linear inequalities [9]. For example, Eq. (1) can be relaxed to three linear inequalities as follows

$$L \cdot Z_a(k) \le V_a(k - l_a/w_a^b) + k_a \cdot l_a - U_a(k) \le \le U \cdot (1 - Z_a(k)) + \varepsilon \tag{11a}$$

$$L \cdot Z_a(k) \le X_a(k) - S_a^{in} \le U \cdot Z_a(k) \tag{11b}$$

$$L \cdot (1 - Z_a(k)) \le X_a(k) - v_a(k - l_a/w_a^b) \le U \cdot (1 - Z_a(k)) \tag{11c}$$

Where L, U are very large negative and positive constants, respectively. ε is a very small positive constant, $Z_a(k)$ is a binary variable (0,1) indicating "if-then" conditions of eq.(1). $Z_a(k) = 1$ for the case $V_a(k - l_a/w_a^b) + k_a \cdot l_a \le U_a(k)$; $Z_a(k) = 0$ otherwise. Substituting the two values of $Z_a(k)$ into eq. (11a), one can see.

$$Z_a(k) = 1 \Rightarrow L \le V_a(k - l_a/w_a^b) + k_a \cdot l_a - U_a(k) \le \le \varepsilon \Rightarrow V_a(k - l_a/w_a^b) + k_a \cdot l_a \le U_a(k) \tag{12a}$$

$$Z_a(k) = 0 \Rightarrow 0 \le V_a(k - l_a/w_a^b) + k_a \cdot l_a - U_a(k) \le \le U + \varepsilon \Rightarrow V_a(k - l_a/w_a^b) + k_a \cdot l_a > U_a(k) \tag{12b}$$

Then the value of $Z_a(k)$ can further determine the possible inflow $X_a(k)$. If $Z_a(k) = 0$ (1), eqs. (11b-11c), become:

$$Z_a(k) = 1 \Rightarrow \begin{cases} L \le X_a(k) - S_a^{in} \le U \\ 0 \le X_a(k) - v_a(k - l_a/w_a^b) \le 0 \end{cases} \Rightarrow \Rightarrow X_a(k) = v_a(k - l_a/w_a^b) \tag{13a}$$

$$Z_a(k) = 0 \Rightarrow \begin{cases} 0 \le X_a(k) - S_a^{in} \le 0 \\ L \le X_a(k) - v_a(k - l_a/w_a^b) \le U \end{cases} \Rightarrow \Rightarrow X_a(k) = S_a^{in} \tag{13b}$$

This completes the verification.

Further, eq. (4) can be relaxed to the following linear inequalities

$$S_{a_i}^{out} - U \cdot F_{a_i}^1(k) \le v_{a_i}(k) \le S_{a_i}^{out} \tag{14a}$$

$$u_{a_i}(k - l_{a_i}/w_{a_i}^f) + q_{a_i}(k-1)/\delta - U \cdot F_{a_i}^2(k) \le \le v_{a_i}(k) \le u_{a_i}(k - l_{a_i}/w_{a_i}^f) + q_{a_i}(k-1)/\delta \tag{14b}$$

$$\lambda_{b,a_i}(k) \cdot X_b(k) - U \cdot F_{a_i}^3(k) \le \le v_{a_i}(k) \le \lambda_{b,a_i}(k) \cdot X_b(k) \tag{14c}$$

$$F_{a_i}^1(k) + F_{a_i}^2(k) + F_{a_i}^3(k) = 2 \tag{14d}$$

where, $F_{a_i}^1(k)$, $F_{a_i}^2(k)$, $F_{a_i}^3(k)$ are binary variables which must be either one or zero. U is a very large positive constant. According to eq. (14d), in the three binary variables, only one variable is 0, the other two variables must be 1. If $F_{a_i}^1(k) = 0$, eq. (14a) is equivalent to:

$$F_{a_i}^1(k)=0 \Rightarrow \begin{cases} S_{a_i}^{out} \le v_{a_i}(k) \le S_{a_i}^{out} \\ u_{a_i}(k-l_{a_i}/w_{a_i}^f)+q_{a_i}(k-1)/\delta - U \le \\ \quad \le v_{a_i}(k) \le \\ \quad \le u_{a_i}(k-l_{a_i}/w_{a_i}^f)+q_{a_i}(k-1)/\delta \\ \lambda_{b,a_i}(k)\cdot X_b(k) - U \le v_{a_i}(k) \le \\ \quad \le \lambda_{b,a_i}(k)\cdot X_b(k) \end{cases} \Rightarrow$$
$$\Rightarrow v_{a_i}(k)=S_{a_i}^{out}$$

According to the above explanation, if $F_{a_i}^1(k)=0$, the actual link exit flow is equal to $S_{a_i}^{out}$. Summarized, eq. (4) can be replaced by these four linear constraints eqs. (14a-14d).

Finally, the mixed integer linear program based SO-DTA formulations are presented

Objective function: eq. (10)

Constraints: eqs. (3, 6, 7, 8, 9, 11a, 11b, 11c, 14a, 14b, 14c, 14d)

The total amount of flow, however, out of a diverging node during time interval k is determined by objective function (5). This flow could be less than the flow suggested by the program; however, the total turning flow during time interval k is beneficial for the overall system wide travel. After relaxation, the formulation becomes a mixed integer linear program and can be solved easily by general linear programming solvers. We use the linear programming solver to solve it.

4. EXAMPLES

The example network is a 6-link, two-OD network illustrated in *Figure 1*. The dotted lines represent the dummy link relative to the origin and destination. The entry, exit capacity and maximum storage space on link 1 are 1,500veh/h, 500veh/h, 1,000veh, respectively. The entry, exit capacity and maximum storage space of other links are 3,000veh/h, 1,000veh/h, 1,000veh, respectively. The total time interval is 2h. Other parameters: $l_a/w_a^f=0.2$h, $l_a/w_a^b=0.4$h, $\delta=0.1$h.

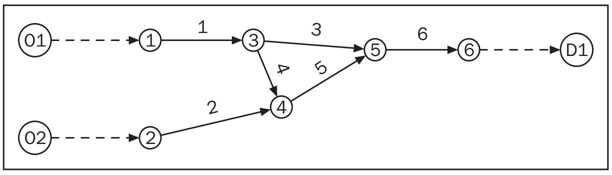

Figure 1 - Example network

Firstly, set $q_i(k)=1{,}000; 3\le k\le 6; i=\{1,2\}$. *Figure 2* gives the flow states on link 1. Because the exit capacity is only 500veh/h, the queue appears from time interval 5, and the link exit flow is equal to the exit capacity. Although the model can obtain the queue phenomenon, the link flow backup propagation does not appear.

Further, set $q_i(k)=3{,}000; 3\le k\le 6; i=\{1,2\}$. *Figure 3* gives the flow states on link 1. At time interval 5, the inflow on link 1 is equal to the entry capacity. Then, as the queue increases, the link queue starts to back up the link entry, and the link inflow at time interval 11 is 500veh/h. We can find the link inflow may change over time since the queue physically backs up and may reduce the link inflow of the upstream link.

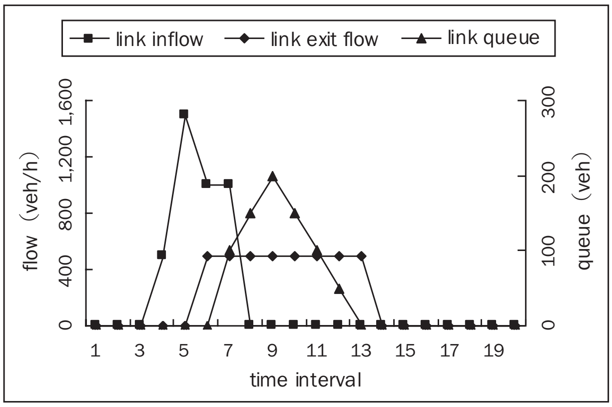

Figure 2 - The flow on link 1 (demand 1)

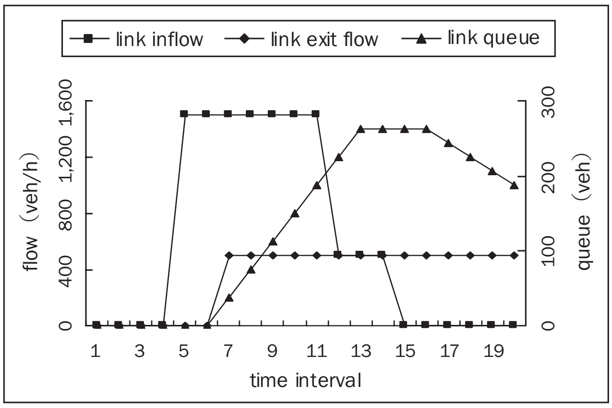

Figure 3 - The flow on link 1 (demand 2)

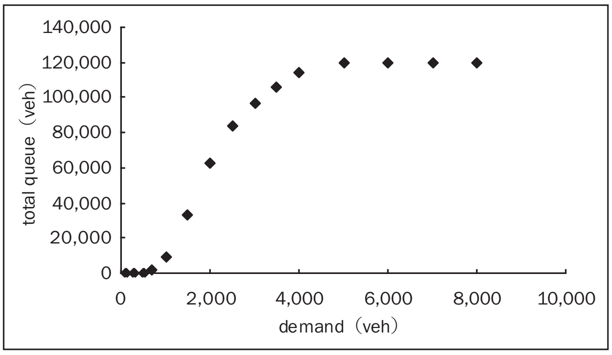

Figure 4 - The change of total queue

Figure 4 gives the change of the total queue with the increasing demands. X Coordinates represent the demands between each O-D pair between time intervals 3 and 6, Y Coordinates represent the evolution of the queue length. When the demands are smaller than 500, the queue length is zero; when the demands are larger than 500, the queue can appear and increase with the increasing demands; however, when the demands are larger than 5,000, the total queue length remains unchanged since the link storage space limits the entering network demand. When the demands are larger than a value, all links in the network are congested, the surplus demand does not enter the network. In other words, the network is in the state of maximum saturation. However, the case cannot occur under the point queue concept, because the queue can increase with the increase of the demand.

5. CONCLUSIONS

A mixed integer linear program for the single destination system optimum dynamic traffic assignment problem with physical queue is presented. In order to model the spillback queue on the link, one assumes the flow states consist of two cases: one case is uncongested, all vehicles can run according to the free flow speed; another is congested, the link capacity may change over time since a queue physically backs up and may reduce the link exit capacity of the upstream link. The linearization method is used in relaxing the non-linear equations of the model. Then the general linear software is adopted to solve it. In a simple example, the model can reflect the link flow propagation and queue backup and so on. And we find the total queue length will remain unchanged when the road network is saturated.

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