XIN WANG
E-mail: bitimath@163.com
Beijing University of Aeronautics and Astronautics
School of Economics and Management
Beijing 100191, China
Beijing Information Science and Technology University
School of Applied Science
Beijing 100192, China
HAI-JUN HUANG
E-mail: haijunhuang@buaa.edu.cn
Beijing University of Aeronautics and Astronautics
School of Economics and Management
Beijing 100191, China

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BI-CRITERIA SYSTEM OPTIMUM TRAFFIC ASSIGNMENT IN NETWORKS WITH CONTINUOUS VALUE OF TIME

ABSTRACT

For an elastic demand transportation network with continuously distributed value of time, the system disutility can be measured either in time units or in cost units. The user equilibrium model and the system optimization model are each formulated in two different criteria. The conditions required for making the system optimum link flow pattern equivalent to the user equilibrium link flow pattern are derived. Furthermore, a bi-objective model has been developed which minimizes simultaneously the system travel time and the system travel cost. The existence of a pricing scheme with anonymous link tolls which can decentralize a Pareto system optimum into the user equilibrium has been investigated.

KEY WORDS

elastic demand, value of time, bi-criteria, user equilibrium, Pareto optimum

1. INTRODUCTION

Roadway congestion is a source of enormous economic costs. In principle, many of these costs can be prevented because they result from socially inefficient choices made by individuals. Traffic congestion has been alleviated in a number of regions by introducing congestion pricing. Indeed, road pricing has become one of the priorities on transport policy agendas all over the world. It is increasingly believed that road pricing may offer an effective instrument to manage travel demand, and to raise revenues that may for instance be used for transport improvements. In recent years, more and more pricing schemes have been proposed, tested or implemented worldwide [1].

In the research of congestion pricing, the conventional user equilibrium models suppose that all travellers have the same value of time (VOT), i.e., the road network users are homogenous. In the case of homogeneous users, the theory of marginal cost pricing has well been established in general traffic networks. In line with this theory, various investigations have been conducted on how this classical economic principle would work in a general road network with queuing [2] or in a stochastic network [3]. However, because of the difference in income levels, the network users generally have different VOTs and different preferences on travel choices. Therefore, in travel behaviour modelling and analyses, user heterogeneity has to be considered.

The network disutility can be measured in travel time or travel cost. It is obvious that different system optimal (SO) flow patterns will be obtained if we use different units (time or money) to measure the system disutility. There exists a transformation relationship between time disutility and cost disutility due to VOTs.

In the presence of user heterogeneity, various equilibrium traffic assignment models have been formulated in different ways. The deterministic formulae with discrete VOTs were developed by, for example, Leurent [4], Mayet and Hansen [5], Nagurney [6,7]. The optimal pricing problem for heterogeneous users with continuous VOTs was studied by Dial [8, 9]. The multi-criteria or cost-versus-time network equilibrium and the system optimum problem in a network with a discrete set of VOT user classes were examined by

Yang and Huang [10]. They proved that there exists a uniform link toll scheme that supports a cost-based system optimum as a multi-class user equilibrium (UE) flow pattern.

In some networks, there are oligopoly Cournot–Nash (CN) firms. Users in a CN firm can collaborate together for minimizing the total cost of the firm and then compete against others. Zhang, Yang and Huang [11] examined such a UE–CN mixed equilibrium. They showed that in a network with both UE and CN users, applying the conventional marginal-cost pricing in order to reach a system optimum requires all link tolls be differentiated across user classes. Yu and Huang investigated the upper bound of the efficiency loss of such UE-CN mixed equilibrium when all link cost functions are polynomial [12].

Recently, Guo and Yang [13] have proven that any Pareto optimum can be decentralized into multi-class user equilibrium by using positive anonymous link tolls in the fixed demand network with discrete VOTs. They further bounded the system performance gap when optimized by two different criteria. For an elastic demand network with discrete VOTs, Clark et al. showed that there also exists anonymous link toll scheme which can decentralize the cost-based SO flows to UE flows [14]. Wang and Huang showed that any Pareto optimum, except the time-based one, can be decentralized into a multi-class user equilibrium by a pricing scheme with positive anonymous tolls on all links in the networks with discrete VOTs and elastic demand [15].

In this paper, a new model has been developed in which two criteria for system optimization are taken into account simultaneously through expressing the welfare functions in either monetary or time units. The VOT is a continuous distribution among a population of commuters and the travel demand is elastic. In Sections 2 and 3, the network equilibrium and the system optimum are formulated, each in time and monetary units, respectively. Then, the equivalent conditions between user equilibrium flow pattern and system optimum flow pattern are derived. Section 4 presents a bi-objective system optimization model which minimizes the system time and system cost simultaneously. The existence of anonymous link tolls which can drive the Pareto optimum to a UE solution is investigated. Section 5 concludes the paper.

2. TIME-BASED AND COST-BASED EQUILIBRIA

Let $G=(V,A)$ be a graph, where V is a set of nodes and A a set of arcs. This graph can be taken to represent a transportation network where nodes are the intersections and arcs are the links between intersections. In this network, there is a set of origin-destination (OD) pairs, W, and a set of paths, R, connecting OD pairs. All network users differ by the value of times (VOTs) and the probability distribution of VOT value, β, across users is defined by a function of distribution: $F(x)=P(\beta\leq x)$. Let the considered population be ordered in a decreasing order of their VOTs and $\beta(v)$ be the VOT of the v$^{\text{th}}$ user. We assume $\beta(v)>0$ is continuous and differentiable function on its domain of definition and $\beta'(v)<0$, although it is in fact a discrete probability space. From this definition, we have $\beta(v)=F^{-1}(1-v/N)$ since $(N-v)/N=F(\beta(v))$, where N is the total number of considered users. The travel demand q_w from an origin to a destination is a function of the travel time between OD pair $w\in W$. This travel demand function is assumed to be invertible. The travel time of link a, $t_a(v_a)$, is a function of the flow v_a on link a, $a\in A$, and assumed to be strictly monotonically increasing. Let R_w be the set of all simple paths connecting OD pair $w\in W$, f_{rw} the flow on path r connecting OD pair w. The flow on link a can be expressed in terms of path flows, i.e.,

$$v_a=\sum_{w\in W}\sum_{r\in R_w}f_{rw}\delta_{ar},$$

where δ_{ar} is 1 if link a is on path r and zero otherwise, R_w is the set of paths connecting OD pair w.

Let Ω denote the feasible set of all path flows,

$$\Omega=\left\{(\mathbf{f},\mathbf{q}):\sum_{r\in R}f_{rw}=q_w,f_{rw}\geq 0,q_w\geq 0,r\in R_w,w\in W\right\}, \tag{1}$$

where $\mathbf{f}=(\cdots,f_{rw},\cdots)^{\mathrm{T}}$ is the vector of path flows and $\mathbf{q}=(\cdots,q_w,\cdots)^{\mathrm{T}}$ the vector of OD demands.

Generally, the flow pattern under the system optimum is different from the UE flow pattern because each user in UE makes the decision of path choice only to minimize the private travel cost rather than the social cost. This says that all users in UE do not consider the influence of their decisions on the whole system. Thus, congestion pricing is introduced to guide the people's behaviour for implementing a SO flow pattern in a UE manner.

When all users are assigned onto links, we still assume that the VOT function of link flows is continuously distributed. Let $\beta_a(v)$ and $\beta_w(v)$ be the VOT of the v^{th} user, corresponding to link $a\in A$ and OD pair $w\in W$, respectively. The generalized cost of the v^{th} user on path r between OD pair w in the time unit is

$$c_{rw}^{t}=\sum_{a\in A}\left(t_a(v_a)+\frac{u_a}{\beta_a(v)}\right)\delta_{ar},\ r\in R_w,\ w\in W, \tag{2}$$

where u_a is the toll charged on link a. In this study, only the anonymous and non-negative link tolls have been considered, because the negative and discriminative link tolls are difficult to implement in reality. The generalized cost of the v^{th} user on path r between OD pair w can also be given in monetary unit, i.e.,

$$c_{rw}^{c}=\sum_{a\in A}(\beta_a(v)t_a(v_a)+u_a)\delta_{ar},\ r\in R_w,\ w\in W. \tag{3}$$

Let $\mathbf{u}=(\cdots,u_a,\cdots)^{\mathrm{T}}$ be the vector of all link tolls.

Under the scheme of charging all link users, the time-based user equilibrium (TUE) with elastic demand is the solution of the following optimization problem:

$$\min_{(\mathbf{f},\mathbf{q})\in\Omega} Z(\mathbf{f},\mathbf{q}) = \min_{(\mathbf{f},\mathbf{q})\in\Omega}\left(\sum_{a\in A}\int_0^{v_a} t_a(x)\mathrm{d}x + \right.$$

$$\left. + \sum_{a\in A}\int_0^{v_a} u_a \frac{1}{\beta_a(v)}\mathrm{d}v - \sum_{w\in W}\int_0^{q_w} g_w(v)\mathrm{d}v\right), \tag{4}$$

where

$$v_a = \sum_{w\in W}\sum_{r\in R_w} f_{rw}\delta_{ar}.$$

The inversed demand function of OD pair $w\in W$, $q_w(q)$, is measured in time unit and assumed to be decreasing. The first-order optimality conditions of the above optimization problem are as follows:

$$\sum_{a\in A} t_a(v_a)\delta_{ar} + \sum_{a\in A}\frac{u_a}{\beta_a(v_a)}\delta_{ar} = \mu_w^t,$$

$$\text{if } f_{rw} > 0,\ r\in R_w, w\in W, \tag{5a}$$

$$\sum_{a\in A} t_a(v_a)\delta_{ar} + \sum_{a\in A}\frac{u_a}{\beta_a(v_a)}\delta_{ar} \geq \mu_w^t,$$

$$\text{if } f_{rw} = 0,\ r\in R_w,\ w\in W, \tag{5b}$$

$$\mu_w^t = g_w(q_w),\ \text{if } q_w > 0,\ w\in W, \tag{5c}$$

$$\mu_w^t \geq g_w(q_w),\ \text{if } q_w = 0,\ w\in W, \tag{5d}$$

where μ_w^t is the multiplier associated with the flow conservation equation

$$\sum_{r\in R} f_{rw} = q_w$$

and $\beta_a(v_a)$ is the VOT of the last user on link $a\in A$. Equations (5a)-(5d) clearly define a time-based user equilibrium (UE) about route choice and having or not having the travel. In this equilibrium, μ_w^t can be regarded as the minimal travel disutility of OD pair *w*, i.e., $\mu_w^t = \min_{r\in R_w}\{c_{rw}^t\}$ in the time unit.

In the TUE problem (4), $\beta_a(v)$, $t_a(v_a)$ and $g_w(q_w)$ are differentiable and monotonic. Hence, the following properties are tenable:

$$\frac{\partial^2 Z}{\partial v_a^2} = \sum_{a\in A} t_a'(v_a) - \sum_{a\in A}\frac{u_a}{(\beta_a(v_a))^2}\beta_a'(v_a) > 0,\ a\in A,$$

$$\frac{\partial^2 Z}{\partial q_w^2} = -g_w'(q_w) > 0,\ w\in W.$$

Therefore, the objective function $Z(\mathbf{f},\mathbf{q})$ is strictly convex with respect to link flow vector $\mathbf{v} = (\cdots, v_a, \cdots)^{\mathrm{T}}$,

$$v_a = \sum_{w\in W}\sum_{r\in R_w} f_{rw}\delta_{ar},$$

and OD demand vector $\mathbf{q}$. Then, the solution of link flow and OD demand is unique at the equilibrium state in time unit. Note that the objective function is not strictly convex subject to path flow $\mathbf{f}$, so the solution of path flow is not unique.

The cost-based (or monetary-based) user equilibrium (CUE) with elastic demand can be obtained by solving the following optimization problem:

$$\min_{(\mathbf{f},\mathbf{q})\in\Omega} \tilde{Z}(\mathbf{f},\mathbf{q}) = \min_{(\mathbf{f},\mathbf{q})\in\Omega}\left(\sum_{a\in A}\int_0^{v_a}\beta_a(v)t_a(v)\mathrm{d}v + \right.$$

$$\left. + \sum_{a\in A} u_a v_a - \sum_{w\in W}\int_0^{q_w}\beta_w(v)g_w(v)\mathrm{d}v\right). \tag{6}$$

The first-order optimality conditions of the problem (6), or the equilibrium conditions of the CUE are as follows:

$$\sum_{a\in A}\beta_a(v_a)t_a(v_a)\delta_{ar} + \sum_{a\in A} u_a\delta_{ar} = \mu_w^c,$$

$$\text{if } f_{rw} > 0,\ r\in R_w,\ w\in W, \tag{7a}$$

$$\sum_{a\in A}\beta_a(v_a)t_a(v_a)\delta_{ar} + \sum_{a\in A} u_a\delta_{ar} \geq \mu_w^c,$$

$$\text{if } f_{rw} = 0,\ r\in R_w,\ w\in W, \tag{7b}$$

$$\mu_w^c = \beta_w(q_w)g_w(q_w),\ \text{if } q_w > 0,\ w\in W, \tag{7c}$$

$$\mu_w^c \geq \beta_w(q_w)g_w(q_w),\ \text{if } q_w = 0,\ w\in W, \tag{7d}$$

where μ_w^c is the multiplier associated with equation

$$\sum_{r\in R} f_{rw} = q_w$$

and $\beta_w(q_w)$ is the VOT of the last traveller between OD pair $w\in W$. In the equilibrium, μ_w^c can be regarded as the minimal travel disutility in monetary unit, i.e., $\mu_w^c = \min_{r\in R_w}\{c_{rw}^c\}$.

Similarly, we can prove that the objective function of problem (6) is convex subject to OD demand $\mathbf{q}$, due to

$$\frac{\partial^2\tilde{Z}}{\partial q_w^2} = -(\beta_w'(q_w)g_w(q_w) + \beta_w(q_w)g_w'(q_w)) > 0,\ w\in W.$$

Thus, the solution of link flow and OD demand is also unique at the equilibrium state in monetary unit.

3. SYSTEM OPTIMA

First, the cost-based (or monetary-based) system optimum (CSO) was formulated:

$$\min_{(\mathbf{f},\mathbf{q})\in\Omega} C(\mathbf{f},\mathbf{q}) = \min_{(\mathbf{f},\mathbf{q})\in\Omega}\left(\sum_{a\in A}\int_0^{v_a}\beta_a(v)t_a(v_a)\mathrm{d}v - \right.$$

$$\left. - \sum_{w\in W}\int_0^{q_w}\beta_w(v)g_w(v)\mathrm{d}v\right), \tag{8}$$

where $C(\mathbf{f},\mathbf{q})$ is the system travel disutility measured in monetary unit, which equals the total social cost of travelling minus the total user benefit from travelling.

The first-order optimality conditions of the program (8) are as follows:

$$\sum_{a\in A}\left(\beta_a(v_a)t_a(v_a) + \frac{\mathrm{d}t_a(v_a)}{\mathrm{d}v_a}\int_0^{v_a}\beta_a(v)\mathrm{d}v\right)\delta_{ar} = \mu_w^{so,c},$$

$$\text{if } f_{rw} > 0,\ r\in R_w,\ w\in W, \tag{9a}$$

$$\sum_{a\in A}\left(\beta_a(v_a)t_a(v_a) + \frac{\mathrm{d}t_a(v_a)}{\mathrm{d}v_a}\int_0^{v_a}\beta_a(v)\mathrm{d}v\right)\delta_{ar} \geq \mu_w^{so,c},$$

$$\text{if } f_{rw} = 0,\ r\in R_w,\ w\in W, \tag{9b}$$

$$\mu_w^{so,c} = \beta_w(q_w)g_w(q_w),\ \text{if } q_w > 0,\ w\in W, \tag{9c}$$

where $\beta_a(v_a)$ is the VOT of the last user on link $a \in A$ and $\beta_w(q_w)$ is the VOT of the last traveller between OD pair $w \in W$. Let

$$u_a = (\mathrm{d}t_a(v_a)/\mathrm{d}v_a)\int_0^{v_a}\beta_a(v)\mathrm{d}v$$

be the link toll. Clearly, the optimality conditions (9a)-(9d) for the CSO can be regarded as the network equilibrium conditions in cost unit. The multiplier $\mu_w^{so,c}$ is the corresponding minimal OD travel cost in monetary unit. Hence, we have the following Theorem 1.

Theorem 1 *There exists a link tolling scheme* **u** *to decentralize the cost-based system optimum flow pattern into a user equilibrium link flow pattern in cost unit.* □

Note that the link toll can be rewritten as

$$u_a = \frac{\mathrm{d}t_a(v_a)}{\mathrm{d}v_a}\int_0^{v_a}\beta_a(v)\mathrm{d}v = \frac{\mathrm{d}t_a(v_a)}{\mathrm{d}v_a}v_a\frac{\int_0^{v_a}\beta_a(v)\mathrm{d}v}{v_a},$$

where $v_a\mathrm{d}t_a(v_a)/\mathrm{d}v_a$ is the externality measured in time unit. The externality is the additional travel time that a marginal user imposes on other users who have already travelled on link $a \in A$. This externality is identical for all users.

$$\int_0^{v_a}\beta_a(v)\mathrm{d}v/v_a$$

is the average VOT of all users traversing the link. Thus, the link toll u_a is irrelevant to $\beta_a(v)$ of the v^{th} user, but determined by the link users' VOT distribution and the link flow.

The time-based system optimum (TSO) can be formulated as

$$\min_{(\mathbf{f},\mathbf{q})\in\Omega} T(\mathbf{f},\mathbf{q}) = \min_{(\mathbf{f},\mathbf{q})\in\Omega}\left(\sum_{a\in A}t_a(v_a)v_a - \sum_{w\in W}\int_0^{q_w}g_w(v)\mathrm{d}v\right). \tag{10}$$

The first-order optimality conditions of program (10) are as follows:

$$\sum_{a\in A}t_a(v_a)\delta_{ar} + \sum_{a\in A}v_a\frac{\mathrm{d}t_a(v_a)}{\mathrm{d}v_a}\delta_{ar} = \mu_w^{so,t},\quad \text{if } f_{rw} > 0,\ r\in R_w,\ w\in W, \tag{11a}$$

$$\sum_{a\in A}t_a(v_a)\delta_{ar} + \sum_{a\in A}v_a\frac{\mathrm{d}t_a(v_a)}{\mathrm{d}v_a}\delta_{ar} \geq \mu_w^{so,t},\quad \text{if } f_{rw} = 0,\ r\in R_w,\ w\in W, \tag{11b}$$

$$\mu_w^{so,t} = g_w(q_w),\ \text{if } q_w > 0,\ w\in W, \tag{11c}$$

$$\mu_w^{so,t} \geq g_w(q_w),\ \text{if } q_w = 0,\ w\in W. \tag{11d}$$

Comparing (5a)-(5d) and (11a)-(11d), we can find that if letting $u_a = \beta_a(v_a)v_a(\mathrm{d}t_a(v_a)/\mathrm{d}v_a)$ in (5a), the optimality conditions of TSO are equivalent to the equilibrium conditions of TUE. Thus, we have the following Theorem 2.

Theorem 2 *Under a discriminatory link tolling scheme, a user equilibrium link flow pattern is equivalent to the system optimum in time unit.* □

For achieving the system optimum in time unit, each user travelling on a link must bear a marginal social travel time which consists of a marginal private travel time and a travel time externality. The toll on link $a\in A$ is $u_a = \beta_a(v_a)v_a(\mathrm{d}t_a(v_a)/\mathrm{d}v_a)$, herein $v_a(\mathrm{d}t_a(v_a)/\mathrm{d}v_a)$ is just the time externality. Since the link toll is the product of the time externality and the VOT of the marginal user, hence, it is discriminatory and unrealistic.

4. BI-CRITERIA PARETO SYSTEM OPTIMUM

As mentioned before, both system time and system cost can be used to measure the system performance. Nevertheless, previous studies on network pricing typically consider only the minimization of either system time T or system cost C, or study the minimization of T and C simultaneously but with fixed demand. In this paper, we propose a bi-objective minimization problem with elastic demand, which combines the two SO problems (8) and (10) together. Instead of seeking optimal flow patterns of minimizing total system time or total system cost, this bi-objective problem is to seek a Pareto optimal solution set. At each point of the Pareto optimal solution set, neither T nor C can be further reduced without increasing the other one. We have tried to find a non-negative link tolling scheme which can decentralize a Pareto SO flow pattern into the UE state. The bi-objective minimization problem is

$$\min_{(\mathbf{f},\mathbf{q})\in\Omega}\begin{Bmatrix}T(\mathbf{f},\mathbf{q})\\ C(\mathbf{f},\mathbf{q})\end{Bmatrix} = \min_{(\mathbf{f},\mathbf{q})\in\Omega}\begin{Bmatrix}\sum_{a\in A}t_a(v_a)v_a - \sum_{w\in W}\int_0^{q_w}g_w(v)\mathrm{d}v\\ \sum_{a\in A}\int_0^{v_a}\beta_a(v)t_a(v_a)\mathrm{d}v - \sum_{w\in W}\int_0^{q_w}\beta_w(v)g_w(v)\mathrm{d}v\end{Bmatrix}, \tag{12}$$

where

$$v_a = \sum_{w\in W}\sum_{r\in R_w}f_{rw}\delta_{ar}$$

as defined in previous sections.

4.1 Decentralization of a given feasible target link flow pattern

For a given feasible target link flow $\bar{\mathbf{v}} = (\cdots,\bar{v}_a,\cdots)^{\mathrm{T}}$, the following non-linear programming problem is considered:

$$\min_{(\mathbf{f},\mathbf{q})\in\Omega} C(\mathbf{f},\mathbf{q}) = \min_{(\mathbf{f},\mathbf{q})\in\Omega}\left(\sum_{a\in A}\int_0^{v_a}\beta_a(v)t_a(\bar{v}_a)\mathrm{d}v - \sum_{w\in W}\int_0^{q_w}\beta_w(v)g_w(v)\mathrm{d}v\right) \tag{15}$$

subject to

$$\sum_{w\in W}\sum_{r\in R_w}f_{rw}\delta_{ar} = \bar{v}_a,\ a\in A. \tag{16}$$

The Lagrange function of program (15)-(16) is

$$L(\mathbf{f},\mathbf{q},\lambda,\boldsymbol{\mu}) = \sum_{a\in A}\int_0^{\bar{v}_a}\beta_a(v)t_a(\bar{v}_a)\mathrm{d}v -$$

$$-\sum_{w\in W}\int_0^{q_w}\beta_w(v)g_w(v)\mathrm{d}v + \sum_{a\in A}\lambda_a\left(\bar{v}_a - \sum_{w\in W}\sum_{r\in R_w} f_{rw}\delta_{ar}\right) +$$

$$+\sum_{w\in W}\mu_w\left(q_w - \sum_{r\in R_w} f_{rw}\right). \tag{17}$$

The optimality conditions of program (15)-(16) are

$$\sum_{a\in A}\beta_a(\bar{v}_a)t_a(\bar{v}_a)\delta_{ar} - \sum_{a\in A}\lambda_a\delta_{ar} - \mu_w \geq 0,$$

$$r\in R_w,\ w\in W, \tag{18}$$

$$f_{rw}\left(\sum_{a\in A}(\beta_a(\bar{v}_a)t_a(\bar{v}_a) - \lambda_a)\delta_{ar} - \mu_w\right) = 0,$$

$$r\in R_w,\ w\in W, \tag{19}$$

$$-\beta_w(q_w)g_w(q_w) + \mu_w \geq 0,\ w\in W, \tag{20}$$

$$q_w(-\beta_w(q_w)g_w(q_w) + \mu_w) = 0,\ w\in W, \tag{21}$$

where $\lambda = (\cdots,\lambda_a,\cdots)^{\mathrm{T}}$ and $\boldsymbol{\mu} = (\cdots,\mu_w,\cdots)^{\mathrm{T}}$ are the multipliers associated with the equality constraints (16) and the OD flow conservation constraint (1), respectively. Note that here λ_a and μ_w are unrestricted in sign.

It can be observed that if letting $u_a = -\lambda_a$ to be the anonymous link toll, the optimality conditions (18)-(21) are simply the CUE conditions (7a)-(7d), with μ_w being regarded as μ_w^c. This equivalence between the CSO optimality conditions and the CUE conditions immediately leads to the following Theorem 3.

Theorem 3 *Any feasible target link flow pattern can be supported as a cost-based user equilibrium by imposing anonymous link tolls which may be positive or negative.*

Proof. *For any feasible link flow $\bar{\mathbf{v}} = (\cdots,\bar{v}_a,\cdots)^{\mathrm{T}}$, we can construct the CSO problem (15)-(16). Suppose $(\mathbf{f},\mathbf{q})$ is an optimal solution satisfying the optimality conditions (18)-(21). These conditions are just the CUE conditions (7a)-(7d) under the tolling scheme $u_a = -\lambda_a$, thus, $(\mathbf{f},\mathbf{q})$ is a cost-based user equilibrium flow pattern and $\bar{\mathbf{v}}$ is the corresponding link flow. Because λ_a is unrestricted in sign, the link toll u_a may be positive or negative.* □

Theorem 1 states that under any anonymous link tolling scheme, users with higher VOT tend to choose routes with lower travel time by paying higher toll charges. Thus, the system cost C which is exclusive of toll charge and given as the sum of the travel times of all users weighted by their VOTs, is naturally minimized.

4.2 Existence of non-negative anonymous link tolls

For the bi-objective SO problem (12), we now prove that there exists a non-negative anonymous link tolling scheme to implement a Pareto system optimum in a UE manner.

Lemma 1 *A Pareto system optimum can be supported as a cost-based user equilibrium by implementing an anonymous link tolling scheme if the tolling scheme can produce the aggregate link flow of the Pareto system optimum.*

Proof. *Suppose $(\bar{\mathbf{f}},\bar{\mathbf{q}})$ is a Pareto optimum of the bi-objective SO problem (12), and let $\bar{\mathbf{v}}$ be the corresponding link flow pattern. From Theorem 3, there exists an anonymous link toll scheme $\mathbf{u}$ under which the feasible link pattern $\bar{\mathbf{v}}$ can be decentralized in a CUE manner. Let $(\mathbf{f},\mathbf{q})$ be the corresponding cost-based user equilibrium. It can reduce the system cost because of the equivalence of (7a)-(7d) and (18)-(21), i.e., $C(\mathbf{f},\mathbf{q}) \leq C(\bar{\mathbf{f}},\bar{\mathbf{q}})$.*

Furthermore, let

$$\mu_w^t = \min_{r\in R_w}\left(\sum_{a\in A}(t_a(\bar{v}_a) + u_a/\beta_a(v_a))\delta_{ar}\right)$$

be the minimal travel disutility in time unit. Then $g_w(\bar{q}_w) \geq \mu_w^t$. Moreover, since $\mathbf{q}$ is the equilibrium demand, then $\mu_w^t \geq g_w(q_w)$ from (7c) and (7d). Thus, we have $g_w(\bar{q}_w) \geq g_w(q_w)$, which simply means $\bar{q}_w \leq q_w$ from the monotonicity of inversed demand function. Hence, we have

$$T(\mathbf{f},\mathbf{q}) = \sum_{a\in A}\bar{v}_a t_a(\bar{v}_a) - \sum_{w\in W}\int_0^{q_w} g_w(v)\mathrm{d}v \leq$$

$$\leq \sum_{a\in A}\bar{v}_a t_a(\bar{v}_a) - \sum_{w\in W}\int_0^{\bar{q}_w} g_w(v)\mathrm{d}v = T(\bar{\mathbf{f}},\bar{\mathbf{q}}).$$

This concludes $T(\mathbf{f},\mathbf{q}) = T(\bar{\mathbf{f}},\bar{\mathbf{q}})$ and $C(\mathbf{f},\mathbf{q}) = C(\bar{\mathbf{f}},\bar{\mathbf{q}})$ because $(\bar{\mathbf{f}},\bar{\mathbf{q}})$ is a Pareto optimum. Thus, $(\bar{\mathbf{f}},\bar{\mathbf{q}})$ is a solution of the cost-based user equilibrium. □

Theorem 3 and Lemma 1 ensure the existence of anonymous link tolls which can induce a Pareto system optimum. The existence of non-negative anonymous link tolls is now proven.

Theorem 4 *A Pareto system optimum can be supported as the cost-based user equilibrium by non-negative anonymous link tolls.*

Proof. *Suppose $(\bar{\mathbf{f}},\bar{\mathbf{q}})$ is a Pareto optimum of the bi-objective SO problem (12). From Lemma 2, there exists an anonymous link tolling scheme $\mathbf{u}$ under which $(\bar{\mathbf{f}},\bar{\mathbf{q}})$ can be the CUE solution. We construct a modified version of model (15) in which the equality constraint (16) is replaced by the following inequality constraint:*

$$\sum_{w\in W}\sum_{r\in R_w} f_{rw}\delta_{ar} \leq \bar{v}_a,\ a\in A. \tag{22}$$

Because constraint (22) is a simple relaxation of constraint (16) and other constraints remain unchanged, the feasible region of the new model is expanded and hence non-empty. This means that the new model has solutions. Suppose the solution of the new model is $(\mathbf{f},\mathbf{q})$ and $(\boldsymbol{\lambda},\boldsymbol{\mu})$ is the Lagrange multiplier of the constraint. Because $(\mathbf{f},\mathbf{q})$ satisfies the optimality conditions (18)-(21) and conditions (7a)-(7d) required by the cost-based user equilibrium, we then have $\bar{\mathbf{q}}=\mathbf{q}$ due to the uniqueness of the cost-based equilibrium demand. It is known that the multiplier associated with a 'less than' inequality like (22) is negative at the solution point. So, $\mathbf{u}=-\lambda$ is non-negative.

It can be proven that the constraint (22) is binding at the solution point. Suppose (22) is not binding at the optimal solution $(\mathbf{f},\mathbf{q})$. Then, the corresponding link flow of $\mathbf{f}$ is such $\mathbf{v}$ that $\mathbf{v}\leq\bar{\mathbf{v}}$ holds. Let $v_b<\bar{v}_b$ for a specific link $b\in A$. Then we have $t_b(v_b)<t_b(\bar{v}_b)$ and $t_b(v_b)v_b<t_b(\bar{v}_b)v_b$ by the monotonicity of the link cost function. This leads to

$$\sum_{a\in A}t_a(v_a)v_a-\sum_{w\in W}\int_0^{q_w}g_w(v)\mathrm{d}v<$$

$$<\sum_{a\in A}t_a(\bar{v}_a)\bar{v}_a-\sum_{w\in W}\int_0^{q_w}g_w(v)\mathrm{d}v, \tag{23}$$

$$\sum_{a\in A}\int_0^{v_a}\beta_a(v)t_a(v_a)\mathrm{d}v\leq\sum_{a\in A}\int_0^{\bar{v}_a}\beta_a(v)t_a(\bar{v}_a)\mathrm{d}v. \tag{24}$$

Note that (23) is simply the relation $T(\mathbf{f},\mathbf{q})<T(\bar{\mathbf{f}},\mathbf{q})$. From (24), we have

$$\sum_{a\in A}\int_0^{v_a}\beta_a(v)t_a(v_a)\mathrm{d}v-\sum_{w\in W}\int_0^{q_w}\beta_w(v)g_w(v)\mathrm{d}v\leq$$

$$\leq\sum_{a\in A}\int_0^{\bar{v}_a}\beta_a(v)t_a(\bar{v}_a)\mathrm{d}v-\sum_{w\in W}\int_0^{q_w}\beta_w(v)g_w(v)\mathrm{d}v. \tag{25}$$

Clearly, (25) means $C(\mathbf{f},\mathbf{q})\leq C(\bar{\mathbf{f}},\mathbf{q})=C(\bar{\mathbf{f}},\bar{\mathbf{q}})$. It contradicts the assumption that $(\bar{\mathbf{f}},\bar{\mathbf{q}})$ is a Pareto optimum. This completes the proof. □

The implication of Theorem 4 is that, when considering simultaneous minimization of system time and system cost, there exits a non-negative common link toll scheme that supports a Pareto system optimum as cost-based user equilibrium. However, the common link toll in this case is not based on the user externality, but determined from the Lagrange multiplier of a dual programming problem.

5. CONCLUSION

In reality, the users having different VOTs in a traffic network are willing to experience different travel burdens, that is, the users are inclined to minimize either individual travel cost or individual travel time. Meanwhile, the government would like to maximize the system total social welfare or minimize the system total travel cost. Hence, the system performance can be measured either in time unit or in cost unit. They are two significantly different criteria for evaluation. These two criteria are in general mutually inconsistent with each other.

This paper provided a theoretical investigation of the multi-criteria traffic assignment, including the cost-versus-time network equilibrium and system optimum in a network with continuous VOT distribution and elastic demand. We have formulated the user equilibrium and system optimum each in time unit and monetary unit. We have proven the existence of link tolling schemes that can decentralize an SO link flow pattern into the UE link flow pattern in two different criteria, respectively. Under the monetary criterion, the link toll is equal to the average VOT of all users traversing that link multiplied by the travel time externality. On the other hand, when the system disutility is measured in time unit, the link toll is the product of the travel time externality and the VOT of a marginal user.

Furthermore, a bi-objective model has been developed to minimize the system travel time and system travel cost simultaneously. We have proven that there exists a pricing scheme with non-negative anonymous link tolls which can decentralize a Pareto system optimum into user equilibrium. Moreover, the non-negative anonymous link toll can be determined from the Lagrange multiplier of a dual programming problem.

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摘要

连续时间价值分布网络中的双准则系统最优交通分配

对于用户时间价值连续分布的弹性需求交通网络，系统效能既可以用时间测定也可以用金钱测定。本文首先分别建立两种测度单位下的用户均衡和系统最优交通分配模型，推导使两种交通分配规则所导致流量分布等价的条件，然后提出一个同步令系统总时间和系统总成本最小的双目标优化模型，并证明存在一个匿名收费制度，使帕累托系统通过用户均衡的方式实现。

关键词

弹性需求，时间价值，双准则，用户均衡，帕累托最优

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