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Science in Traffic
Original Scientific Paper
U. D. C.: 519.21:62-1
Accepted: Feb. 21, 2000
Approved: Dec. 19, 2001

A GENERALISED STOCK-OUT FUNCTION FOR CONTINUOUS PRODUCTION VARIABLES

This paper was presented at the Second Mini-Symposium on MRP, Laplace Transforms and Input-Output Analysis, Bovec, Slovenia, February 13-17, 1999

ABSTRACT

This paper follows up a previous study for optimising a production-inventory system when external demand is stochastic. A modified stock-out function is presented to cover more general situation when cumulative production may be a continuous variable. Optimisation equations are further investigated, including the sufficient conditions for optimisation.

KEY WORDS

stock-out function, stochastic demand, inventory

1. INTRODUCTION

The stock-out function for a renewal process has been introduced in several papers [1, 2], with the assumption that cumulative production has an integer value. However, this assumption also creates a limitation in the optimisation conditions for the production plan [3, 4], namely that we need to use the inequality in the first-order differences with respect to the decision variables concerning cumulative production.

These optimisation conditions have been used to study safety stock problem for production planning. One important finding is that the level of safety stock has a linear relation to the square root of time [5]. In order to be confident of this result, we believe there is a necessity to further investigate the integer value assumption in the model and the sufficient conditions for optimisation.

In this paper, we first release the mentioned integral assumption and develop a modified stock-out function. Then we investigate whether the optimal production conditions need to be adjusted based on this modification. Finally we present some comments and draw our conclusion.

2. STOCK-OUT FUNCTION

In our study, renewal processes are used to describe the stochastic properties of the production-inventory system. A renewal demand process is such that demand arrives by unit events separated by stochastic time intervals, all with identical independent probability distribution function pdf $f(t)$, $t \geq 0$. One theorem [cf. 2] regarding this process is that the stock-out function (expectation of stock-outs) in the Laplace frequency domain follows

$$E(\tilde{B}(s))=\frac{\tilde{f}^{\overline{P}+1}}{s(1-\tilde{f})}, \tag{1}$$

where $\overline{P}$ is an integer and refers to the cumulative production and $\tilde{f}$ is the transform of the pdf. For a demand renewal process, we assume that the time interval of demand events is a continuous variable and demand is a discrete variable. According to [1], the transform of the probability that cumulative demand $\overline{D}$ equals j at time t is

$$\mathcal{L}\{\Pr(\tilde{D}(t)=j)\}=\tilde{f}^{j}\,(1-\tilde{f})/s, \tag{2}$$

where j is an integer, since demand always arrives as units. Because $1/s$ corresponds to the integration over time, Equation (2) is interpreted as the difference of two cumulative distribution functions. For the sake of removing the integer value assumption, we use the following definition for stock-outs B

$$\begin{cases} B=\overline{D}-\overline{P}, & \text{for } \overline{D}\geq\overline{P} \\ B=0, & \text{for } \overline{D}<\overline{P} \end{cases} \tag{3}$$

where $\overline{D}$ remains to be an integer, $\overline{P}$ and B are any non-negative numbers. We then obtain

$$\mathcal{L}\{\Pr(B=x)\}=\mathcal{L}\{\Pr(\overline{D}=x+\overline{P})\}=\frac{\tilde{f}^{x+\overline{P}}(1-\tilde{f})}{s}, \tag{4}$$

Furthermore, the sum $x+\overline{P}$ needs to be an integer and x, $\overline{P}$ are any non-negative numbers.

$$\mathrm{E}(\tilde{B}(s))=\sum_{x=0}^{\infty}\frac{\tilde{f}^{x+\overline{P}}(1-\tilde{f})}{s}. \tag{5}$$

This summation takes place over the instances of x, when $x+\overline{P}$ is integral. If we substitute the above equation with $\begin{cases} x=j+\varepsilon \\ \overline{P}=\overline{P'}-\varepsilon \end{cases}$, where j and $\overline{P'}$ are integers and $0 \le \varepsilon < 1$, we have

$$E(\widetilde{B}(s))=\sum_{x\ge0}^{\infty}\frac{\widetilde{f}^{x+\overline{P}}(1-\widetilde{f})}{s}=.$$

$$=\sum_{j=0}^{\infty}(j+\varepsilon)\cdot\frac{\widetilde{f}^{j+\overline{P'}}(1-\widetilde{f})}{s}=$$

$$=\frac{\widetilde{f}^{\overline{P'}}(1-\widetilde{f})}{s}\sum_{j=0}^{\infty}j\cdot\widetilde{f}^{j}+\frac{\widetilde{f}^{\overline{P'}}(1-\widetilde{f})}{s}\sum_{j=0}^{\infty}\widetilde{f}^{j}=$$

$$=\frac{\widetilde{f}^{\overline{P'}+1}}{s(1-\widetilde{f})}+\frac{\varepsilon\widetilde{f}^{\overline{P'}}}{s}. \qquad (6)$$

We notice now that the expected stock-out function consists of two parts. The first part is the same as the previous stock-out function and the second part concerns essentially the transform of a cumulative probability distribution for a sum of $\overline{P'}$ intervals having an additional coefficient ε. The expression in (6) is linear in ε, when ε varies in the interval [0, 1].

For instance, for a Poisson process, the stock-out function in its time domain, which is the inverse of the above expression, should be written as

$$E(B(t))=\lambda t-$$
$$-\overline{P'}\sum_{j=0}^{\overline{P'}-1}\frac{(\lambda t)^j}{j!}(\overline{P'}-j)+\varepsilon(1-e^{-\lambda t}\sum_{j=0}^{\overline{P'}-1}\frac{(\lambda t)^j}{j!}) \qquad (7)$$

When cumulative production is an integral number, ε is zero and we obtain

$$E(B(t))=\lambda t-\overline{P}+\sum_{j=0}^{\overline{P}-1}\frac{(\lambda t)^j}{j!}(\overline{P}-1) \qquad (8)$$

which gives the same result as in [4]

3. DERIVATIVES AND DIFFERENCES OF STOCK-OUT FUNCTION

The properties of the stock-out function have been discussed in previous papers. However, little has concerned cases when cumulative production $\overline{P}$ is considered as a variable. This section attempts to make a contribution from this aspect.

The first-order derivative of $E(\widetilde{B}(t))$ with respect to $\overline{P}$ is

$$\frac{\partial E(\widetilde{B}(t))}{\partial\overline{P}}=\begin{cases}-\dfrac{\partial E(\widetilde{B}(s))}{\partial\varepsilon}=-\dfrac{\widetilde{f}^{\overline{P'}}}{s}, \text{ for } \overline{P'}-1<\overline{P}<\overline{P'} \\ \text{does not exist, for } \overline{P} \text{ integer}\end{cases}$$

(9)

which shows that the stock-out function is continuous in the open interval (P' – 1, $\overline{P'}$) and piecewise continuous in its domain. We define the difference of $E(\widetilde{B}(s))$ with respect to the integer $\overline{P}$ as

$$\Delta E(\widetilde{B}(s))_{\overline{P}}=E(\widetilde{B}(s))_{\overline{P}+1}-E(\widetilde{B}(s))_{\overline{P}}. \qquad (10)$$

Therefore

$$\Delta E(\widetilde{B}(s))_{\overline{P}}=\frac{\widetilde{f}^{\overline{P}+1+1}}{s(1-\widetilde{f})}-\frac{\widetilde{f}^{\overline{P}+1}}{s(1-\widetilde{f})}=-\frac{\widetilde{f}^{\overline{P}+1}}{s}. \qquad (11)$$

This first-order difference brings a similar result as the first-order derivative but has a different interpretation. For the second-order derivatives of $E(\widetilde{B}(s))$ with respect to $\overline{P}$, we obtain the following expression

$$\frac{\partial^2 E(\widetilde{B}(t))}{\partial\overline{P}^2}=\begin{cases}0, \text{ for } \overline{P'}-1<\overline{P}<\overline{P'} \\ \text{does not exist, for } \overline{P} \text{ integer}\end{cases} \qquad (12)$$

and similarly

$$\Delta^2 E(\widetilde{B}(s))_{\overline{P}}=\Delta E(\widetilde{B}(s))_{\overline{P}+1}-\Delta E(\widetilde{B}(s))_{\overline{P}}=$$
$$=-\frac{\widetilde{f}^{\overline{P}+1+1}}{s}+\frac{\widetilde{f}^{\overline{P}+1}}{s}=\frac{\widetilde{f}^{\overline{P}+1}(1-\widetilde{f})}{s}. \qquad (13)$$

This apparently can be interpreted as the transform of the probability that cumulative demand during time t equals $\overline{P}$ + 1, which is positive. The above derivative and difference expressions show that the expected stock-out function is convex with a negative slope when $\overline{P}$ is varied.

4. INTEGRAL VALUES FOR OPTIMISATION CONDITIONS

The stock-out function over the whole planning horizon, where characteristic function needs to be incorporated due to different levels of cumulative production $\overline{P}$, k = 1, 2, ..., is given by

$$E(\widetilde{B}(s))=\frac{1}{2\pi i}\sum_{k=0}^{n}\int_{w=\beta-i\infty}^{\beta+i\infty}E(\widetilde{B}(w))\cdot$$
$$\cdot\frac{e^{-(s-w)t_k}-e^{-(s-w)t_{k+1}}}{s-w}dw. \qquad (14)$$

We then have the following results

$$\frac{E(\widetilde{B}(s))}{\partial\overline{P}_k}=-\frac{1}{2\pi i}\int_{w=\beta-i\infty}^{\beta+i\infty}\frac{\widetilde{f}^{\overline{P}_k}}{w}\cdot$$
$$\cdot\frac{e^{-(\rho-w)t_k}-e^{-(\rho-w)t_{k+1}}}{\rho-w}dw, \text{ for } \overline{\text{P}} \text{ not an integer}, \qquad (15)$$

$$\Delta(\widetilde{B}(s))_{\overline{P}_k}=-\frac{1}{2\pi i}\int_{w=\beta-i\infty}^{\beta+i\infty}\frac{\widetilde{f}^{\overline{P}_k}}{w}\cdot$$
$$\cdot\frac{e^{-(\rho-w)t_k}-e^{-(\rho-w)t_{k+1}}}{\rho-w}dw, \text{ for } \overline{\text{P}} \text{ an integer}, \qquad (16)$$

$$\frac{\partial \mathrm{E}(\widetilde{B}(s))}{\partial t_k}=e^{-st_k}\left[[\mathrm{E}(B(t_k))]_{\overline{P}_{k-1}}-[\mathrm{E}(B(t_k))]_{\overline{P}_k}\right],$$

for any $\overline{P}_{k-1}$, $\overline{P}_k$ (17)

When we study the optimal production plan problem, the following objective function is used

$$\mathrm{NPV}=r[\mathrm{E}(\widetilde{D}(\rho))-\rho\mathrm{E}(\widetilde{B}(\rho))]-$$

$$-\sum_{k=1}^{n}(K+c(\overline{P}_k-\overline{P}_{k-1}))e^{-\rho t_k} \tag{18}$$

which counts the cash inflow of sales (r = revenue per unit), the delay payment of backlogging and the cash outflow of production costs (K = setup cost, c = unit production cost). Based on this objective function developed above, the following necessary conditions for optimisation are obtained

$$F_t=\frac{\partial \mathrm{NPV}}{\partial t_k}=\rho e^{-\rho t_k}[-r[\mathrm{E}(B(t_k))]_{\overline{P}_{k-1}}-$$

$$-[\mathrm{E}(\mathrm{B}(t_k))]_{\overline{P}_k}]+(K+c(\overline{P}_k-\overline{P}_{k-1}))]=0 \tag{19}$$

$$F_{\overline{P}}=\frac{\partial \mathrm{NPV}}{\partial \overline{P}_k}=r\rho(\frac{1}{2\pi i}\int_{w=\beta-i\infty}^{\beta+i\infty}\frac{\widetilde{f}^{\overline{P}_k}}{w}\cdot$$

$$\cdot\frac{e^{-(\rho-w)t_k}-e^{-(\rho-w)t_{k+1}}}{\rho-w}dw)-$$

$$-c(e^{-\rho t_k}-e^{-\rho t_{k+1}})=0 \tag{20}$$

or

$$F_{\overline{P}}=\Delta\mathrm{NPV}=r\rho(\frac{1}{2\pi i}\int_{w=\beta-i\infty}^{\beta+i\infty}\frac{\widetilde{f}^{\overline{P}_k}}{w}\cdot$$

$$\cdot\frac{e^{-(\rho-w)t_k}-e^{-(\rho-w)t_{k+1}}}{\rho-w}dw)-$$

$$-c(e^{-\rho t_k}-e^{-\rho t_{k+1}})\leq 0 \tag{21}$$

According to our discussion of the stock-out function in the previous section, we can conclude that the first part of $F_{\overline{P}}$ is monotonically decreasing as a function of $\overline{P}$. It has jumps when $\overline{P}$ takes on integer values and remains constant otherwise. The magnitude of each jump depends on the shape of the probability function $\Pr\{\overline{D}(t)=\overline{P}+1\}$. Figure 1 illustrates the situation. The following limits are obtained:

$$\lim_{\overline{P}\to 0}F_{\overline{P}}=\lim_{\overline{P}\to 0}r\rho(\frac{1}{2\pi i}\int_{w=\beta-i\infty}^{\beta+i\infty}\frac{\widetilde{f}^{\overline{P}_k}}{w}\cdot$$

$$\cdot\frac{e^{-(\rho-w)t_k}-e^{-(\rho-w)t_{k+1}}}{\rho-w}dw)-$$

$$-c(e^{-\rho t_k}-e^{-\rho t_{k+1}})=(r-c)(e^{-\rho t_k}-e^{-\rho t_{k+1}})>0 \tag{22}$$

$$\lim_{\overline{P}\to 0}F_{\overline{P}}=\lim_{\overline{P}\to 0}r\rho(\frac{1}{2\pi i}\int_{w=\beta-i\infty}^{\beta+i\infty}\frac{\widetilde{f}^{\overline{P}_k}}{w}\cdot$$

$$\cdot\frac{e^{-(\rho-w)t_k}-e^{-(\rho-w)t_{k+1}}}{\rho-w}dw)-$$

$$-c(e^{-\rho t_k}-e^{-\rho t_{k+1}})=-c(e^{-\rho t_k}-e^{-\rho t_{k+1}})<0 \tag{23}$$

The above limits exist when t_k and t_{k+1} are finite values.

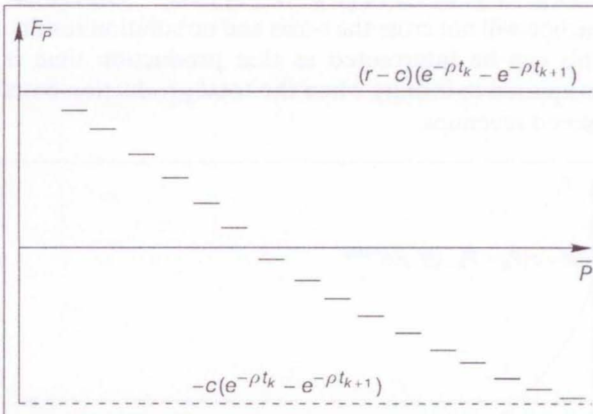

Figure 1 - $F_{\overline{P}}$ curve where $\overline{P}$ is considered as a continuous variable, and t_k and t_{k+1} are constants

Therefore, the solution of $F_{\overline{P}}=0$ is either unique and an integral number $\overline{P}$ is obtained (Equation 21 is satisfied), or is a set of values where $\overline{P}'-1\leq\overline{P}\leq\overline{P}'$ (both equations 20 and 21 are satisfied). In the latter case $F_{\overline{P}}=0$ and the line is overlapped by the $\overline{P}$ axis. Since time is a continuous variable, we consider this as a very exceptional situation that seldom occurs.

On the other hand, the F_t function has the following derivative

$$\frac{\partial F_t}{\partial t}=-r\rho e^{-\rho t_k}\left[\mathcal{L}^{-1}\left\{\frac{\widetilde{f}^{\overline{P}+1}}{1-\widetilde{f}}\right\}_{\overline{P}_{k-1}}-\mathcal{L}^{-1}\left\{\frac{\widetilde{f}^{\overline{P}+1}}{1-\widetilde{f}}\right\}_{\overline{P}_k}\right]=-r\left[\mathcal{L}^{-1}\left\{\sum_{j=\overline{P}_{k-1}}^{\overline{P}_{k-1}}\widetilde{f}^{j}\right\}\right]<0 \tag{24}$$

Since $\widetilde{f}(0)=1$ and $\widetilde{f}(\infty)=0$ (if $f(0)<\infty$), we obtain the following boundaries using the initial value and final value theorems

$$\lim_{t\to\infty}F_t=\lim_{s\to 0}\rho e^{-\rho t_k}\left[-r\left\{\sum_{j=\overline{P}_{k-1}}^{\overline{P}_{k-1}}\widetilde{f}^{j}\right\}+K+\right.$$

$$\left.+c(\overline{P}_k-\overline{P}_{k-1})\right]=(-r(\overline{P}_k-\overline{P}_{k-1})+K+$$

$$+c(\overline{P}_k-\overline{P}_{k-1}))\cdot\rho e^{-\rho t_k} \tag{25}$$

$$\lim_{t\to\infty}F_t=\lim_{s\to 0}\rho e^{-\rho t_k}\left[-r\left\{\sum_{j=\overline{P}_{k-1}}^{\overline{P}_{k-1}}\widetilde{f}^{j}\right\}+K+\right.$$

$$\left.+c(\overline{P}_k-\overline{P}_{k-1})\right]=(K+c(\overline{P}_k-\overline{P}_{k-1}))\cdot e^{-\rho t_k} \tag{26}$$

$$\lim_{t\to\infty}\frac{\partial F_t}{\partial t}=\lim_{s\to 0}\left(-\rho e^{-\rho t_k} r\left[s\left\{\sum_{j=\bar{P}_{k-1}}^{\bar{P}_{k-1}}\tilde{f}^{j}\right\}\right]\right)=0 \quad (27)$$

The F_t curve approaches $-r(\bar{P}_k-\bar{P}_{k-1})+K+$ $+c(\bar{P}_k-\bar{P}_{k-1})$ when time is made infinite and its slope then tends towards zero (Figure 2). It is shown that the solution of $F_t = 0$ is unique if and only if $-r(\bar{P}_k-\bar{P}_{k-1})+K+c(\bar{P}_k-\bar{P}_{k-1})\le 0$. Otherwise, the line will not cross the t-axis and no solution results. This can be interpreted as that production time is postponed to infinity when the total production costs exceed revenues.

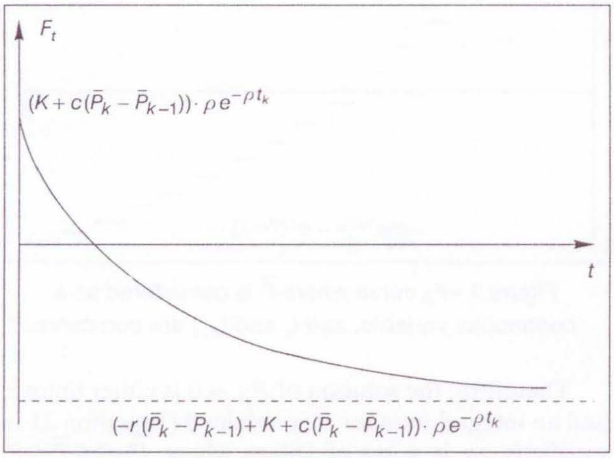

Figure 2 - F_1 curve where t is considered as a continuous variable, and $\bar{P}_k$ and $\bar{P}_{k-1}$ as constants

Figures 1 and 2 indicate that the iterative solving procedure in [4] generates a unique solution in each step. However, it does not guarantee a global optimal for the final solution.

5. STRUCTURE OF THE HESSIAN MATRIX FOR OPTIMISATION

Throughout the paper using the same methodology for optimising the production plan, only the necessary conditions for optimisation have been presented. In this section we investigate the structure of the Hessian matrix for optimisation. The second-order derivatives (differences) of the objective function are obtained as

$$A_k=\frac{\partial^2\mathrm{NPV}}{\partial t_k^{\,2}}=-r\rho e^{-\rho t_k}\left[\mathcal{L}^{-1}\left\{\frac{\tilde{f}^{\bar{P}+1}}{1-\tilde{f}}\right\}_{\bar{P}_{k-1}}-\mathcal{L}^{-1}\left\{\frac{\tilde{f}^{\bar{P}+1}}{1-\tilde{f}}\right\}_{\bar{P}_k}\right]=-r\rho e^{-\rho t_k}\left[\mathcal{L}^{-1}\left\{\sum_{j=\bar{P}_{k-1}}^{\bar{P}_{k-1}}\tilde{f}^{j}\right\}\right]<0 \quad (28)$$

$$B_k=\Delta^2(\tilde{B}(s))_{\bar{P}_k}=-r\rho\left(\frac{1}{2\pi i}\int_{w=\beta-i\infty}^{\beta+i\infty}\frac{\tilde{f}^{\bar{P}_k+1}(1-\tilde{f})}{w}\cdot\frac{e^{-(\rho-w)t_k}-e^{-(\rho-w)t_{k+1}}}{\rho-w}\,dw\right)<0 \quad (29)$$

$$a_k=\Delta\left(\frac{\partial\mathrm{NPV}}{\partial t_k}\right)_{\bar{P}_k}=$$

$$=-r\rho e^{-\rho t_k}\left[\mathcal{L}^{-1}\left\{\frac{\tilde{f}^{\bar{P}_k}}{s}\right\}\right]+c\rho e^{-\rho t_k} \quad (30)$$

$$b_k=\Delta\left(\frac{\partial\mathrm{NPV}}{\partial t_k}\right)_{\bar{P}_{k-1}}=$$

$$=-r\rho e^{-\rho t_k}\left[\mathcal{L}^{-1}\left\{\frac{\tilde{f}^{\bar{P}_{k-1}}}{s}\right\}\right]-c\rho e^{-\rho t_k} \quad (31)$$

Other second-order derivatives (differences) are all zeros. Let n be the optimal total number of batches in the planning horizon. Hence there are $2n$ decision variables and we have a $2n\times 2n$ dimension Hessian matrix with the following structure

$$\begin{bmatrix}
A_1 & & & a_1 & & & \\
& A_2 & & b_1 & a_2 & & \\
& & \ddots & & \ddots & \ddots & \\
& & A_n & & & b_n & a_n \\
a_1 & b_1 & & B_1 & & & \\
& a_2 & \ddots & & B_2 & & \\
& & \ddots & b_n & & \ddots & \\
& & & a_n & & & B_n
\end{bmatrix}$$

Elements in its main diagonal all have negative values. The signs of a_k and b_k are undetermined. However, it is still difficult to determine analytically if this matrix is negative definite.

6. SUFFICIENT CONDITION - A FIXED ORDER QUANTITY CASE

In our previous studies, the order quantity for each production batch has been allowed to vary. This apparently enlarges the number of variable and makes analysis consequently more complex. In this section, we concentrate on a special case of early optimisations, namely a fixed order quantity Q is applied at each stage in the production. This assumption also often holds in practice, for instance, when the batch size of a machine needs to be fixed for some technical reason. The objective function therefore needs to be rewritten as

$$\mathrm{NPV}=r\left[\mathrm{E}(\widetilde{D}(\rho))-\rho\mathrm{E}(\widetilde{B}(\rho))\right]-\sum_{k=1}^{n}(K+cQ)e^{-\rho t_k}, \tag{32}$$

where t_k and Q are decision variables. The necessary conditions for optimisation are then

$$F_t=\frac{\partial\mathrm{NPV}}{\partial t_k}=\rho e^{-\rho t_k}\,[-r([\mathrm{E}(B(t_k))]_{(k-1)Q}-$$
$$-[\mathrm{E}(B(t_k))]_{kQ}+K+cQ]=0, \tag{33}$$

$$F_Q=\Delta\mathrm{NPV}_Q=r\rho\sum_{k=0}^{n}\left(\frac{1}{2\pi i}\int_{w=\beta-i\infty}^{\beta+i\infty}\frac{\widetilde{f}^{kQ+1}(1-\widetilde{f}^{k})}{w(1-\widetilde{f})}\cdot\frac{e^{-(\rho-w)t_k}-e^{-(\rho-w)t_{k+1}}}{\rho-w}dw\right)-c\sum_{k=1}^{n}e^{\rho t_k}\le 0 \tag{34}$$

The second-order derivatives and differences are

$$A_k=\frac{\partial^2\mathrm{NPV}}{\partial t_k^{2}}=-r\rho e^{-\rho t_k}\left[\mathcal{L}^{-1}\left\{\sum_{j=(k-1)Q}^{kQ-1}f^{j}\right\}\right]<0 \tag{35}$$

$$B=\Delta F_Q=-r\rho\sum_{k=0}^{n}\frac{1}{2\pi}\int_{w=\beta-i\infty}^{\beta+i\infty}\frac{\widetilde{f}^{kQ+1}(1-f^{k})^2}{w(1-\widetilde{f})}\cdot\frac{e^{-(\rho-w)t_k}-e^{-(\rho-w)t_{k+1}}}{\rho-w}dw<0 \tag{36}$$

$$a_k=\Delta\left(\frac{\partial NPV}{\partial t_k}\right)_Q=\rho e^{-\rho t_k}\left[-r\left([E(B(t_k)]_{(k-1)(Q+1)}-\right.\right.$$
$$\left.\left.-[E(B(t_k)]_{k(Q+1)}\right)+K+c(Q+1)\right] \tag{37}$$

Other second-order derivatives and differences are all zeros.

The Hessian matrix is now

$$\begin{bmatrix} A_1 & & & & a_1\\ & A_2 & & & a_2\\ & & \ddots & & \vdots\\ & & & A_n & a_n\\ a_1 & a_2 & \dots & a_n & B\end{bmatrix}.$$

For a negative definite matrix the signs of the principal minors alternate starting with a negative sign at the left top corner. This is guaranteed until the minor of order $n\times n$ is reached, since the A_i is negative. When going from n to (n+1), we develop

$$\det\begin{bmatrix} A_1 & & & & a_1\\ & A_2 & & & a_2\\ & & \ddots & & \vdots\\ & & & A_n & a_n\\ a_1 & a_2 & \dots & a_n & B\end{bmatrix}=$$
$$=(B-\frac{a_1^2}{A_1}-\frac{a_2^2}{A_2}-\dots-\frac{a_n^2}{A_n})\cdot\det\begin{bmatrix} A_1 & & & \\ & A_2 & & \\ & & \ddots & \\ & & & A_n\end{bmatrix}.$$

The right-hand member has been obtained by adding multiples of each row to the bottom row successively to eliminate the corresponding bottom row elements. The expression in (38) guarantees that the two determinants in the above equation have opposite signs. By taking into account inequality property in Equation 35, Equation 38 is sufficient and necessary conditions for the Hessian matrix to be negative definite.

$$(B-\frac{a_1^2}{A_1}-\frac{a_2^2}{A_2}-\dots-\frac{a_n^2}{A_n})<0 \tag{38}$$

It is also a sufficient condition for maximising the objective function (Equation 32). For an even more specific case where the order quantity is determined externally, the decision variables are only the production batch times t_k. The solution to Equation 33 is then unique and it provides a global optimal, since the Hessian is negative definite.

7. CONCLUSION

This paper has extended the stock-out function to cover a system where cumulative production can be a continuous variable. The results show that the structure of this function is similar to earlier ones when cumulative production is an integer. The properties of this stock-out curve have also been discussed. It is shown that this curve has discontinuous points at the integer $\overline{P}$ positions. Due to the monotonic and jumping characteristics of the optimal condition curve, we conclude that, in general, the optimal solution for $\overline{P}$ needs to contain integer values only. In order to study the sufficient conditions for optimisation, we also have presented the structure of the Hessian matrix. So far no solid conclusion for the general case has been drawn. For very specific cases, in which either the set of cumulative production volumes or the set of batch times are determined externally, the Hessian matrix is negative definite.

SAŽETAK

Ovaj je rad nastavak prethodnih ispitivanja optimizacije sustava proizvodnje - zaliha kada je vanjska potražnja stohastička. Predstavljena je modificirana funkcija nedostatka robe na skladištu koja pokriva općenitije slučajeve kad kumulativna proizvodnja može biti kontinuirana varijabla. Jednadžbe optimizacije se dalje ispituju uključujući i uvjete zadovoljenja optimizacije.

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