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APPLICATION OF TRANSFORMS IN A COMPOUND DEMANDS PROCESS

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ABSTRACT

The compound distribution is of interest for the study of inventory problem, since it provides a more flexible description of the stochastic properties of the system compared to many other approaches such as renewal processes. However, due to the difficulties of obtaining analytical results for the compound distribution, such a type of study is usually limited to searching for a good approximation for replacing the complex model. This paper investigates the possibility to extend a previous stochastic inventory model to cover a compound demand process. Transform methods again play an important role in the analysis for capturing the stochastic properties of the compound distribution.

KEY WORDS

inventory logistics, compound demand, Input-Output analysis.

1. INTRODUCTION

In an inventory system, uncertainty usually comes from more than one source. For instance, a system could face stochastic disturbances both from the demand and supply sides. In this circumstance, a compound distribution is used or needs to be used since it may describe more precisely characteristics of the whole production-inventory system.

A literature review shows that most researchers use a compound distribution to tackle problems in which both demand and lead times are uncertain [1-7]. Some authors [8, 9] have focused on the situation when the number of demand transactions and the size of each demand transaction are stochastic. Others [10, 11] consider all these three factors, lead time, number of demand transactions or size of each transaction, as random variables. Bagchi et al [12] summarised analytical models of compound distribution for modelling production-inventory systems and they further discussed the possible distributions applicable for two categories of demand patterns, namely slow moving and fast moving items [12, 13]. But as many researchers have mentioned, analytical models are usually not tractable and further analysis of the production-inventory system becomes very complex.

The second alternative to deal with a compound distribution is by using approximations and simplifications. For instance, in order to determine the control parameters in an inventory model, the demand during lead time is usually simplified as a normal distribution function and its mean and variance are estimated by calculating the first and the second order moments of the original distribution function. This simplification has at least two disadvantages, namely, it may not fit the actual distribution curve very well and it needs to be truncated for negative demand. Eppen and Martin [3] and Lau [6] pointed out that normalisation may not always be true and the consequent analysis may contain substantial errors. They both provide examples showing how problems occur under the normalisation assumption.

In order to avoid such disadvantages due to approximations, Lau [6] and other researchers [4, 5, 7] adopted other approximation distribution functions, such as the Pearson family of distributions in order to get a better fit of real data. In this case, the moments of distribution need to be calculated up to the forth-order to determine the parameters in the Pearson curve. It is claimed that inventory control parameters such as the reorder point is easily obtained by this approach.

In our previous research [14-18], studies have been focused on a demand renewal process where the intervals between demand events have identical independent distribution functions and demand arrives one unit at a time. Laplace transforms are used to capture the stochastic properties of stockouts and inventories. However, we are also interested in the safety stock problem when demand follows a more erratic pattern such as for a lumpy demand process. Ward [9] indicated that a proper compound distribution is suitable for this purpose.

In the study of stochastic models, the moment-generating function plays an important role. But moment-generating functions have a close relationship with transforms. For instance, non-negative continuous and discrete probability functions associate with the Laplace transform and z-transform respectively.

In this paper we first discuss some basics of compound distributions, especially how the first four moments are calculated using the transformation method. Then we use our previous methodology [cf. 16, 17] to optimise the production plan when demand follows a compound distribution. Other than determining the control parameters such as those in *(s, Q)* and *(s, S)* models, we mainly focus on the behaviour of the stockout function and the optimal safety stock of the optimal production plan.

2. NOTATION

$\mathcal{L}(f(x))=\tilde{f}(s)=\int_0^\infty f(x)\cdot e^{-sx}\,dx$

Laplace transform of any function $f(x)$, where s is the complex Laplace frequency

$\mathcal{L}^{-1}\{\tilde{f}(s)\}=f(x)$

inverse of Laplace transform

$Z\{g_m\}=\hat{g}(z)=\sum_{m=0}^{\infty} z^m g_m$

z-transform of a function g_m with a discrete variable. A more conventional definition is $Z\{g_m\}=\sum_{m=0}^{\infty} z^{-m} g_m$. However, it makes no difference in following studies and our definition provides a more compact expression.

$Z^{-1}\{\hat{g}(z)\}=g_m$

inverse of z-transform

$E[X^k]=\int_{-\infty}^{\infty} x^k f(x)\,dx$

kth *moment* for the distribution with probability density function $f(x)$. Also exists for discrete distribution

$\mu_X=E[X]=\int_{-\infty}^{\infty} x f(x)\,dx$

first-order moment of $f(x)$, also identified as the expectation of $f(x)$

$\mu_{Xk}=E[(X-\mu_X)^k]=\int_{-\infty}^{\infty}(X-\mu_X)^k f(x)\,dx$

k^{th} order *central moment* for the distribution with probability density function $f(x)$, $k=1, 2, \ldots$ $\mu_{X1}=0$. μ_{X2} is the variance and is also written as σ_X^2.

3. TRANSFORMS AND BASICS OF MOMENTS

It is well known that the Laplace transform can be used as a moment-generating function for non-negative stochastic variables

$$\left[\frac{d^n \tilde{f}}{ds^n}\right]_{s=0}=\lim_{s\to 0}\int_0^\infty \frac{d^n}{ds^n}e^{-sx} f(x)\,dx=$$

$$=(-1)^n \lim_{s\to 0}\int_0^\infty x^n e^{-sx} f(x)\,dx=$$

$$=(-1)^n\int_0^\infty x^n f(x)\,dx=(-1)^n E(X^n) \qquad (1)$$

Similarly, for the discrete distribution function g_m with stochastic variable M, we may use the z-transform to generate its moments. Replacing g_m into the above expression and substituting $z=e^{-s}$, we obtain

$$E(M^n)=(-1)^n\lim_{s\to 0}\sum_{m=0}^{\infty}\frac{d^n}{ds^n}(e^{-sm}g_m)=$$

$$=(-1)^n\lim_{s\to 0}\frac{d^n}{ds^n}\left(\sum_{m=0}^{\infty} z^m g_m\right)=(-1)^n\lim_{s\to 0}\frac{d^n\hat{g}}{ds^n}=$$

$$=\lim_{z\to 1} z\cdot\frac{d}{dz}\left(z\cdot\frac{d}{dz}\left(z\cdot\frac{d}{dz}\left(\ldots\left(z\cdot\frac{d\hat{g}}{dz}\right)\right)\right)\right)=$$

$$=\lim_{z\to 1}\frac{d^n\hat{g}}{d(\ln z)^n} \qquad (2)$$

or in a more explicit form

$$Z\{g_m\}\Big|_{z=1}=\sum_{m=0}^{\infty} z^m g_m\Big|_{z=1}=\sum_{m=0}^{\infty} g_m=1 \qquad (3a)$$

$$\frac{dZ\{g_m\}}{dz}\Big|_{z=1}=\sum_{m=0}^{\infty} m z^m g_m\Big|_{z=1}=$$

$$=\sum_{m=0}^{\infty} m g_m=E(M) \qquad (3b)$$

$$\frac{d^2 Z\{g_m\}}{dz^2}\Big|_{z=1}=\sum_{m=0}^{\infty} m(m-1)z^{m-2} g_m\Big|_{z=1}=$$

$$=E(M^2)-E(M) \qquad (3c)$$

$$\frac{d^3 Z\{g_m\}}{dz^3}\Big|_{z=1}=\sum_{m=0}^{\infty} m(m-1)(m-2)z^{m-3} g_m\Big|_{z=1}=$$

$$=E(M^3)-3E(M^2)+2E(M) \qquad (3d)$$

Table 1 - Transforms of some distribution functions

	Probability density function	z-transform	Laplace transform
Binomial $B(n, \beta)$ $0 \le \beta \le 1$ $x = 0, 1, \ldots, n$	$\binom{n}{x} \beta^x (1-\beta)^{n-x}$	$(1-\beta+\beta z)^n$	
Geometric $G(\beta)$ $0 \le \beta \le 1$ $x = 1, 2, \ldots$	$\beta(1-\beta)^{x-1}$	$\frac{\beta z}{1-(1-\beta)z}$	
Poisson $P(\lambda)$ $\lambda > 0$ $x = 0, 1, \ldots$	$e^{-\lambda} \frac{\lambda^x}{x!}$	$e^{\lambda(z-1)}$	
Negative Binominal $NB(r, \beta)$ $0 \le \beta \le 1$ $x = r, r+1, \ldots$	$\binom{x-1}{r-1} \beta^r (1-\beta)^{x-r}$	$\left(\frac{\beta z}{1-(1-\beta)z}\right)^r$	
Exponential $E(\lambda)$ $\lambda > 0$ $x \ge 0$	$\lambda e^{-\lambda x}$		$\frac{\lambda}{\lambda+s}$
Gamma $G(\lambda)$ $\lambda > 0$ $x \ge 0$	$\frac{\lambda^n}{\Gamma(n)} x^{n-1} e^{-\lambda x}$		$\left(\frac{\lambda}{\lambda+s}\right)^n$

$$\left.\frac{d^4 Z\{g_m\}}{d z^4}\right|_{z=1} =$$

$$= \left.\sum_{m=0}^{\infty} m(m-1)(m-2)(m-3) z^{m-4} g_m\right|_{z=1} =$$

$$= \mathrm{E}(M^4) - 6\mathrm{E}(M^3) + 11\mathrm{E}(M^2) - 6\mathrm{E}(M) \quad (3e)$$

....

The transforms for important distributions are summarised in Table 1. The z-transform and the Laplace transform correspond with discrete and continuous distribution respectively. Essentially, discrete distributions can be interpreted as a special case of continuous distributions where impulses exist only at certain positions of equal distance.

The *central moments* are also called the *moments relative to* expectation, which has been defined as $\mu_X = E[X]$. From

$$\mu_{Xk} = \mathrm{E}[(X-\mu_X)^k] = \sum_{j=0}^{k} (-1)^j \binom{k}{j} \mathrm{E}[X^j] \mu_X^{k-j}$$

we obtain the following relation between the central moments and *moments*

$$\mu_{X2} = \mathrm{E}[X^2] - \mathrm{E}[X]^2 \quad (4a)$$

$$\mu_{X3} = \mathrm{E}[X^3] - 3\mathrm{E}[X^2]\mathrm{E}[X] + 2\mathrm{E}[X]^3 \quad (4b)$$

$$\mu_{X4} = \mathrm{E}[X^4] - 4\mathrm{E}[X^3]\mathrm{E}[X] + 6\mathrm{E}[X^2]\mathrm{E}[X]^2 - 3\mathrm{E}[X]^4 \quad (4c)$$

....

Alternatively, the above expressions can be written as

$$\mathrm{E}[X^2] = \mu_{X2} + \mu_X^2 \quad (5a)$$

$$\mathrm{E}[X^3] = \mu_{X3} + 3\mu_X \mu_{X2} + \mu_X^3 \quad (5b)$$

$$\mathrm{E}[X^4] = \mu_{X4} + 4\mu_{X3}\mu_X + 6\mu_X^2 \mu_{X2} + \mu_X^4 \quad (5c)$$

4. BASICS OF THE COMPOUND DISTRIBUTION

The following sum is defined to be the stochastic variable of a compound distribution [cf. 19]

$$W = \sum_{l=1}^{N} Y_i \quad (6)$$

where the Y_i are mutually independent random variables with a common distribution with pdf $f(y)$ and where N also is a random variable with a distribution independent of Y_i and pdf g_n. N and Y_i represent the events and the size of each event respectively. N usually has a discrete distribution and Y_i can be either discrete or continuous. One example of such a compound distribution is the demand in a store where the number of customers and demand from each of them are both independent random variables. A second example is the total demand during the lead time when the lead time and demand rate are both stochastic. The probability is therefore

$$\Pr(w \le W < w + dw) =$$

$$= \sum_{l=1}^{\infty} g_l \cdot \Pr(w \le Y_1 + Y_2 + \ldots + Y_l + < w + dw) =$$

$$=\sum_{l=1}^{\infty} g_l \cdot (f * f ... * f)\mathrm{d}w \quad (7)$$

We notice that the above probability involves the convolution which can be expressed in terms of transform in a compact "compound" way

$$\tilde{w}(s)=\mathcal{L}\{\Pr(w \le W < w+\mathrm{d}w)\}=$$

$$=\sum_{l=1}^{\infty} g_l \cdot \tilde{f}(s)^l = [\mathrm{Z}\{g_l\}]_{z=\tilde{f}(s)} = \hat{g}(\tilde{f}(s)) \quad (8)$$

Using Equation (1, 3 and 4), the above expression generates its moment as

$$\mathrm{E}[W^0]=\tilde{w}(0)=\tilde{g}(\tilde{f}(0))=\tilde{g}(1)=1 \quad (9a)$$

$$\mathrm{E}[W^1]=-\tilde{w}'(0)=\left.\frac{d[\hat{g}(\tilde{f}(s))]}{ds}\right|_{s=0}=$$

$$=-\hat{g}'\cdot\tilde{f}'\big|_{s=0}=\mu_N \mu_Y \quad (9b)$$

$$\mathrm{E}[W^2]=-\tilde{w}^{(2)}(0)=\left.\frac{d^2[\hat{N}(\tilde{Y}(s))]}{ds^2}\right|_{s=0}=$$

$$=-\hat{g}''\cdot(\tilde{f}')^2+\hat{g}'\cdot\tilde{f}''\big|_{s=0}=\mathrm{E}[N^2]\mu_Y{}^2+\mathrm{E}[N]\mu_{Y2} \quad (9c)$$

$$\mathrm{E}[W^3]=-\tilde{w}^{(3)}(0)=-\left.\frac{d^2[\hat{N}(\tilde{Y}(s))]}{ds^2}\right|_{s=0}=$$

$$=-(\hat{g}'''\cdot(\tilde{f}')^3+3\hat{g}''\cdot\tilde{f}'\cdot\tilde{f}''+\hat{g}'\cdot\tilde{f}''')\big|_{s=0}=$$

$$=\mathrm{E}[N^3]\mu_Y{}^3+3\mathrm{E}[N^2]\mu_Y\mu_{Y2}+\mathrm{E}[N]\mu_{Y3} \quad (9d)$$

$$\mathrm{E}[W^4]=-\tilde{w}^{(4)}(0)=\left.\frac{d^4[\hat{g}(\tilde{f}(s))]}{ds^4}\right|_{s=0}=$$

$$=(\hat{g}''''\cdot(\tilde{f}')^4+6\hat{g}'''\cdot(\tilde{f}')^2\cdot\tilde{f}''+$$
$$+3\hat{g}''\cdot(\tilde{f}'')^2+4\hat{g}''\cdot\tilde{f}'\cdot\tilde{f}'''+\hat{g}'\cdot\tilde{f}'''')\big|_{s=0}=$$

$$=\mathrm{E}[N^4]\mu_Y{}^4+6\mathrm{E}[N^3]\mu_Y{}^2\mu_{Y2}+$$
$$+\mathrm{E}[N^2](3\mu_{Y2}{}^2+4\mu_Y\mu_{Y3})+\mathrm{E}[N](\mu_{Y4}-3\mu_{Y2}{}^2) \quad (9e)$$

Replace with Equation (4) and rearrange properly, the above expressions can be written as

$$\mu_W=\mu_N\mu_Y \quad (10a)$$
$$\mu_{W2}=\mu_{N2}\mu_Y{}^2+\mu_N\mu_{Y2} \quad (10b)$$
$$\mu_{W3}=\mu_{N3}\mu_Y{}^3+\mu_N\mu_{Y3}+3\mu_{N2}\mu_Y\mu_{Y2} \quad (10c)$$
$$\mu_{W4}=\mu_{N4}\mu_Y{}^4+6\mu_{N3}\mu_Y{}^2\mu_{Y2}+$$
$$+6\mu_N\mu_{N2}\mu_Y{}^2\mu_{Y2}+3\mu_N{}^2\mu_{Y2}{}^2+3\mu_{N2}\mu_{Y2}{}^2+$$
$$+4\mu_{N2}\mu_Y\mu_{Y3}+\mu_N\mu_{Y4}-3\mu_N\mu_{Y2}{}^2 \quad (10d)$$

The derivation is straightforward here since we have a mixture transform expression in Equation (8). In literature, the derivation up to the fourth-order moments of a compound distribution is only shown in [5, 20] where a different method was adopted. In their approach, first a random variable has been separated into two components and their dependence needs to be clarified. Consequently, an error is easily raised due to the improper assuming of component independence. Nevertheless, our work verifies that only the results in [20] are correct.

The usage of Equation (10) is two-fold. We can either use it to approximate a compound distribution from real data, or for analysing how accurate the simplification is under the circumstance that the individual distributions are known. Both methods are important for investigation properties of a production-inventory model.

5. STOCKOUT AND INVENTORY

At any time, the following relation holds for the stockout and inventory

$$B(t)=\bar{D}(t)-\bar{P}(t)+S(t), \quad (11)$$

where $B(t)$ and $S(t)$ are non-negative numbers and they cannot be both larger than zero at the same time. $\bar{D}$ and $\bar{P}$ are cumulative demand and production respectively. In our study, we are interested in the expected stockout and expected inventory. These two expected values are related to each other as

$$\mathrm{E}(B)=\sum_{j=0}^{\infty} j\cdot\Pr(B=j)=\sum_{j=\bar{P}}^{\infty}(j-P)\cdot\Pr(\bar{D}=j)$$

$$=\sum_{j=0}^{\infty} j\cdot\Pr(\bar{D}=j)-\bar{P}\sum_{j=0}^{\infty}\Pr(\bar{D}=j)+\sum_{j=0}^{\bar{P}-1}(\bar{P}-j)\cdot\Pr(\bar{D}=j)$$

$$=\mathrm{E}(\bar{D})-\bar{P}+\mathrm{E}(S) \quad (12)$$

It is shown that expected stockout and expected inventory are both non-negative numbers and they can both be positive at the same time. Furthermore, in such an inventory system, higher value of expected stockout also means higher value of expected inventory, which is counterintuitive in some sense.

6. THE STOCKOUT FUNCTION FOR A COMPOUND DEMAND PROCESS WITH DEMAND SIZE GEOMETRIC DISTRIBUTED

In general, we know that the safety stock level increases with the variance of demand in a stochastic process. In a compound demand process, the variance of demand increases with time so that its safety stock should not be at constant level as the role it plays in inventory control models such as *(s, Q)* and *(s, S)*. In our previous study [18], a renewal process with Gamma-distributed time interval is used to study how the variance of demand has an impact on safety stock. However, no quantitative conclusions have been drawn.

Now we follow up the same methodology in [16-18] to optimise production planning and search for the safety stock level for different times. This optimal production plan can be basically used as master production schedule (MPS). It is frozen within a certain time interval (planning horizon) when no rescheduling is committed.

We consider a process where the interval of demand transactions has an identical independent distribution function with pdf $f(t)$, or equivalently, where the number of demand transactions is a random variable $N(t)$. In order to make expressions easy to be followed, we let the size in each demand transaction to be a discrete random variable Y_i having a pdf g_m. We also assume that for each demand transaction, the demand should be larger than zero. The probability of demand till time t equals j is

$$\Pr(\bar{D}(t)=0)=\Pr(N(t)=0), \quad \text{for} \quad j=0 \tag{13a}$$

$$\Pr(\bar{D}(t)=j)=\Pr\Big(\sum_i^{N(t)} Y_i=j \Big| N(t)\le j\Big)=$$

$$=\Big(\sum_{l=1}^{l}\Pr N(t)=l\Big)\cdot\Pr\Big(\sum_i^{l} Y_i=j\Big), \quad \text{for} \quad j>0 \tag{13b}$$

It has been proven in [14, 15] that $f(t)$ and $N(t)$ have the relation

$$\mathcal{L}\{\Pr(N(t)=l)\}=\frac{\tilde{f}^l(1-\tilde{f})}{s}, \tag{14}$$

where time t is considered to be a variable and $\tilde{f}$ is the Laplace transform of $f(t)$.

$$\mathcal{L}\{\Pr(\bar{D}(t)=0)\}=\frac{(1-\tilde{f})}{s}, \quad \text{for} \quad l=0 \tag{15a}$$

$$\mathcal{L}\{\Pr(\bar{D}(t)=j)\}=\sum_{l=1}^{j}\frac{\tilde{f}^l(1-\tilde{f})}{s}\cdot\Pr\Big(\sum_{i=1}^{l} Y_i=j\Big)=$$

$$=\sum_{l=1}^{j}\frac{\tilde{f}^l(1-\tilde{f})}{s}\cdot\Pr(Z^{-1}\{\hat{g}(z)^l\}=j), \quad \text{for} \quad l>0 \tag{15b}$$

The complexity of above expression depends on the structure of the inverse of the convolution $\hat{g}(z)$. Without loss generality, a continuous pdf can replace g_m in above expression. However, the inverse of Laplace transform needs to be adopted in this case. Further, we will concentrate on the case when the demand transaction size follows a geometric distribution

$$Y(n)=(1-\beta)^{n-1}\beta, n=1,2,\ldots \tag{16}$$

where β is a non-negative constant less than one. From Table 1, it is found that the inverse of its convolution is a negative binomial distribution

$$\Pr(Z^{-1}\{\hat{g}(z)^l\}=j)=\Pr\left(Z^{-1}\left\{\left(\frac{\beta z}{1-(1-\beta)z}\right)^l\right\}=j\right)=$$

$$=\binom{j-1}{l-1}\beta^l(1-\beta)^{j-l}. \tag{17}$$

We also observe from Table 1 that for the inverse of convolution, Poisson creates Poisson, binomial creates binomial and exponential leads to a Gamma distribution. With the negative binomial distribution in Equation (17), we obtain

$$\mathcal{L}\{\Pr(\bar{D}(t)=j)\}=\sum_{l=1}^{j}\frac{\tilde{f}^l(1-\tilde{f})}{s}\binom{j-1}{l-1}\beta^l(1-\beta)^{j-l}=$$

$$=\frac{\tilde{f}(1-\tilde{f})\beta}{s}\sum_{l=1}^{j}\binom{j-1}{l-1}(\tilde{f}\beta)^{l-1}(1-\beta)^{(j-1)-(l-1)}=$$

$$=\frac{\tilde{f}(1-\tilde{f})\beta}{s}\sum_{l=0}^{j-1}\binom{j-1}{l-1}(\tilde{f}\beta)^{l}(1-\beta)^{(j-1)-l}=$$

$$=\frac{\tilde{f}(1-\tilde{f})\beta}{s}(\tilde{f}\beta+1-\beta)^{j-1} \tag{18}$$

The expected stockout is

$$\mathrm{E}(\tilde{B}(s))=\sum_{j=0}^{\infty} j\cdot\mathcal{L}\{\Pr(D=j+\bar{P})\}=$$

$$=\sum_{j=0}^{\infty} j\frac{\tilde{f}^l(1-\tilde{f})\beta}{s}(\tilde{f}\beta+1-\beta)^{j-l+\bar{P}}=$$

$$=\frac{\tilde{f}(\tilde{f}\beta+1-\beta)^{\bar{P}}}{s(1-\tilde{f})\beta} \tag{19}$$

and the expected demand is

$$\mathrm{E}(\tilde{\bar{D}}(s))=\sum_{j=0}^{\infty} j\cdot\mathcal{L}\{\Pr(\bar{D}=j)\}=\frac{\tilde{f}}{s(1-\tilde{f})\beta}. \tag{20}$$

The final value theorem together with the l'Hôpital's rule show that the long-term difference between the expected cumulative demand and the expected stockout is

$$\lim_{t\to\infty}\mathrm{E}(\bar{D}(t))-\mathrm{E}(B(t))=$$

$$=\lim_{s\to 0} s\cdot\left(\frac{\tilde{f}}{s(1-\tilde{f})\beta}-\frac{\tilde{f}(\tilde{f}\beta+1-\beta)^{\bar{P}}}{s(1-\tilde{f})\beta}\right)=\bar{P}. \tag{21}$$

For the sake of simplicity, we use the expected average cost as our objective function instead of the net present value approach when optimising the production plan. This function consists of three terms of costs, namely the average setup cost, the expected inventory holding cost and the expected stockout cost

$$\mathrm{C}=\frac{K\cdot n}{T}+\frac{h}{T}\cdot\mathrm{E}(\bar{S}(T))+\frac{b}{T}\cdot\mathrm{E}(\bar{B}(t)), \tag{22}$$

where K, h and b are parameters for the setup, holding and stockout costs respectively, n the number of batches and T the planning horizon. We also have the expected stockout during the whole planning horizon as (in time domain)

$$E(\bar{B}(T)) = \sum_{k=0}^{n} E(\bar{\bar{B}}(s)) =$$

$$= \sum_{k=0}^{n} \left[\mathcal{L}^{-1} \left\{ \frac{\tilde{f}(\tilde{f}\beta + 1 - \beta)^{\bar{P}}}{s^2 (1-\tilde{f})\beta} \right\} \right]_{t_k}^{t_{k+1}} . \quad (23)$$

If we take the integral of Equation (12) and rearrange the terms, we obtain the inventory during the whole planning horizon

$$E(\bar{S}(T)) = E(\bar{B}(T)) - E(\bar{\bar{D}}(T)) + E(\bar{\bar{P}}(T)) =$$

$$= \sum_{k=0}^{n} \left[\mathcal{L}^{-1} \left\{ \frac{\tilde{f}(\tilde{f}\beta + 1 - \beta)^{\bar{P}}}{s^2 (1-\tilde{f})\beta} \right\} \right]_{t_k}^{t_{k+1}} -$$

$$-E(\bar{\bar{D}}(T)) + \sum_{k=1}^{n} \bar{P}(t_{k+1} - t_k) \quad (24)$$

For $E(B)$ follows the expression of Equation (19), we have the following necessary optimisation conditions

$$\frac{\partial C}{\partial t_K} = \frac{h}{T}(P_{k-1} - P_k) +$$

$$+ \frac{h+b}{T} \cdot \frac{\partial E(B(\bar{T}))}{\partial t_K} = \frac{h}{T}(P_{k-1} - P_k) +$$

$$+ \frac{h+b}{T} \left[\left[\mathcal{L}^{-1} \left\{ \frac{\tilde{f}(\tilde{f}\beta + 1 - \beta)^{\bar{P}_{k-1}}}{s^2 (1-\tilde{f})\beta} \right\} \right]_{t_k} - \right.$$

$$\left. - \left[\mathcal{L}^{-1} \left\{ \frac{\tilde{f}(\tilde{f}\beta + 1 - \beta)^{\bar{P}_k}}{s(1-\tilde{f})\beta} \right\} \right]_{t_k} \right] = 0 \quad (25)$$

$$\frac{\partial C}{\partial \bar{P}_K} = \frac{h}{T}(t_{k+1} - t_k) + \frac{h+b}{T} \cdot \frac{\partial E(B(\bar{T}))}{\partial \bar{P}_k} =$$

$$= \frac{h}{T}(t_{k+1} - t_k) - \frac{h+b}{T} \cdot \mathcal{L}^{-1} \left\{ \frac{\tilde{f}(\tilde{f}\beta + 1 - \beta)^{\bar{P}_k}}{s^2} \right\} \Bigg|_{t_k}^{t_{k+1}} \geq 0 \quad (26)$$

7. NUMERICAL EXAMPLES

We use the simplest compound demand process, the Stuttering Poisson process [cf. 9 and 21], to demonstrate the above model. A Stuttering Poisson process has the demand size Geometric distributed and the interval of demand an exponential distribution. Therefore

$$\tilde{f} = \frac{\lambda}{\lambda + s}, \quad (27)$$

where λ is the rate of demand. For interval of demand has this type of distribution, the number of demand arrive has a Poisson distribution. Based on the formulas we developed in Section 4, Table 2 summarises the results of the first four moments of a Stuttering Poisson process.

It is already well known that in such a compound Poisson process the mean and variance of cumulative demand increase proportionally with time. Nevertheless, this table also illustrates that the skewness and kurtosis are inversely related with time. When time tends towards infinite, skewness and kurtosis are reduced towards zero and 3, which are the values for a normal distribution. The expressions in the table also show that both skewness and kurtosis are monotonically decreased when β increases. Big value β makes this compound distribution convergent more quickly towards normal distribution. Apparently, when λt and β are small, the approximation of this compound distribution as a normal may create unreliable results.

For the Stuttering Poisson process we have the stockout function in the time domain as

$$E(B) = E(\bar{D}) - \bar{P} + \bar{P} e^{-\lambda t} +$$

$$+ \sum_{j=1}^{\bar{P}} (\bar{P} - j) \sum_{l=1}^{j} \frac{(\lambda t)^l e^{-\lambda t}}{l!} \binom{j-1}{l-1} \beta^l (1-\beta)^{j-l} . \quad (28)$$

The expression of $\dfrac{\partial E(B(T))}{\partial \bar{P}_k}$ is derived as follows

$$\mathcal{L}^{-1}\{\tilde{f}(\tilde{f}\beta + 1 - \beta)^{\bar{P}}\} =$$

$$= \mathcal{L}^{-1} \left\{ \frac{\lambda}{\lambda + s} \left(\frac{\lambda}{\lambda + s}\beta + (1-\beta) \right)^{\bar{P}_k} \right\} =$$

$$= \mathcal{L}^{-1} \left\{ \frac{\lambda}{\lambda + s} \sum_{j=0}^{\bar{P}} \binom{\bar{P}}{j} (1-\beta)^{\bar{P}-j} \beta^j \left(\frac{\lambda}{\lambda + s}\right)^j \right\} =$$

$$= e^{-\lambda t} \sum_{j=0}^{\bar{P}} \binom{\bar{P}}{j} (1-\beta)^{\bar{P}-j} \beta^j \frac{\lambda (\lambda t)^j}{j!} \quad (29)$$

Based on the knowledge that the cumulative probability of a Gamma distribution can be written as the summation of a Poisson distribution, such as

$$\mathcal{L}^{-1} \left\{ \frac{1}{s} \cdot \left(\frac{\lambda}{\lambda + s}\right)^{j+1} \right\} = \int_0^t \frac{\lambda (\lambda \tau)^j}{j!} e^{-\lambda t} d\tau =$$

$$= 1 - e^{-\lambda t} \sum_{i=1}^{j} \frac{(\lambda \tau)^i}{i!} =, \quad (30)$$

then we have

$$\mathcal{L}^{-1} \left\{ \frac{\tilde{f}(\tilde{f}\beta + 1 - \beta)^{\bar{P}_k}}{s} \right\} =$$

$$= \sum_{j=0}^{\bar{P}} \binom{\bar{P}}{j} (1-\beta)^{\overline{P-j}} \beta^j \left(1 - e^{-\lambda t} \sum_{i=0}^{j} \frac{(\lambda t)^i}{i!} \right) \quad (31)$$

Table 2 - Summary of the first four moments of a Stuttering Poisson Process

Distribution	Poisson	Geometric	Stuttering Poisson
Pdf	$P(\lambda t)=e^{-\lambda t}\frac{\lambda t^x}{x!}$ $x=0,1,\ldots$	$G(\beta)=\beta(1-\beta)^{x-1}$ $x=1,2,\ldots$	$s\,P(\lambda t,\beta)=e^{-\lambda t}$, for $x=0$ $s\,P(\lambda t,\beta)=$ $=e^{-\lambda t}\sum_{j=1}^{x}\frac{\lambda t^j}{j!}\cdot\binom{x-1}{j-1}\beta^j(1-\beta)^{x-1}$ $x=1,2,\ldots$
Expectation μ_1	λt	$\frac{1}{\beta}$	$\frac{\lambda t}{\beta}$
2^{nd}-order central moment (variance), μ_2	λt	$\frac{2(1-\beta)}{\beta^2}$	$\frac{\lambda t(2-\beta)}{\beta^2}$
3^{rd}-order central moment, μ_3	λt	$\frac{(1-\beta)(2-\beta)}{\beta^3}$	$\frac{\lambda t(\beta^2-6\beta+6)}{\beta^3}$
skewness[1]	$\frac{1}{\lambda t}$	$\frac{(2-\beta)^2}{1-\beta}$	$\frac{1}{\lambda t}\frac{(\beta^2-6\beta+6)^2}{(2-\beta)^3}$
4^{th}-order central moment, μ_4	$\lambda t+3(\lambda t)^2$	$\frac{(1-\beta)(\beta(\beta-9)+9)}{\beta^4}$	$\frac{3\lambda^2t^2(\beta^2-4\beta+4)^2}{\beta^4}+$ $+\frac{\lambda t(4\beta^2-18\beta+15)}{\beta^4}$
kurtosis[2]	$3+\frac{1}{\lambda t}$	$\frac{\beta(\beta-9)+9}{4(1-\beta)}$	$3+\frac{4}{\lambda t}-\frac{2\beta+1}{\lambda t(2-\beta)^2}$

[1]Skewness = $\frac{\mu_3^2}{\mu_2^3}$; [2]Kurtosis = $\frac{\mu_4}{\mu_2^2}$. Be aware that there are different definitions of skewness and kurtosis.

Take integral one more time, we finally obtain

$$\mathcal{L}^{-1}\left\{\frac{\tilde{f}(\tilde{f}\,\beta+1-\beta)^{\bar{P}}}{s^2}\right\}=$$

$$=\sum_{j=0}^{\bar{P}}\binom{\bar{P}}{j}(1-\beta)^{\bar{P}-j}\beta^j\left(t-\frac{1}{\lambda}\sum_{i=0}^{j}\left[1-e^{-\lambda t}\sum_{l=0}^{i}\frac{(\lambda t)^l}{l!}\right]\right)=$$

$$=\sum_{j=0}^{\bar{P}}\binom{\bar{P}}{j}(1-\beta)^{\bar{P}-j}\beta^j\cdot$$

$$\cdot\left(t-\frac{1}{\lambda}\sum_{i=0}^{j}1+\frac{e^{-\lambda t}}{\lambda}\sum_{i=0}^{j}\sum_{l=0}^{i}\frac{(\lambda t)^l}{l!}\right)=$$

$$=\sum_{j=0}^{\bar{P}}\binom{\bar{P}}{j}(1-\beta)^{\bar{P}-j}\beta^j\cdot$$

$$\cdot\left(t-\frac{l}{\lambda}(j+1)+\frac{e^{-\lambda t}}{\lambda}\sum_{l=0}^{j}(j-l+1)\frac{(\lambda t)^l}{l!}\right) \qquad (32)$$

Equations (25, 26, 27, and 31) constitute the model for optimising the production plan. We use numerical examples to study the safety stock and average cost of the system. Table 3 gives the basic information of the production system.

The mean demand rate is kept constant by fixing the ratio between λ and β. Then we investigate the following cases by changing these two parameters from 0.2 to 0.95. Figures 1 and 2 illustrate that average cost increases with variance and the standard deviation of demand rate. But no apparent quantitative relation has been observed. For demand closer to a lumpy process where β is small and variance is big, the expected costs increase both for holding inventory and stockout.

Table 3 - Parameters for the numerical example

Parameter	Value
planning horizon T	100
mean demand rate λ/β	1
holding cost h	1
backlogging cost b	20, 50
setup cost K	100
batch number n	6

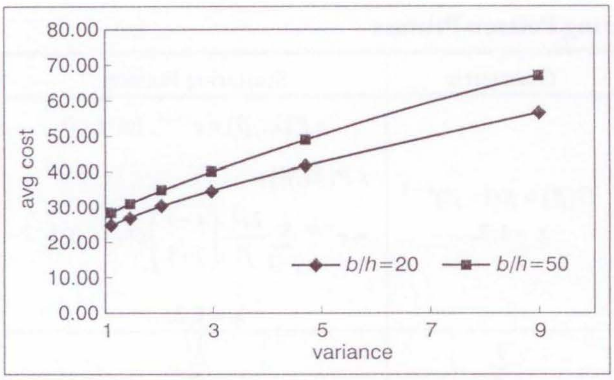

Figure 1 - Average cost vs. variance of demand rate

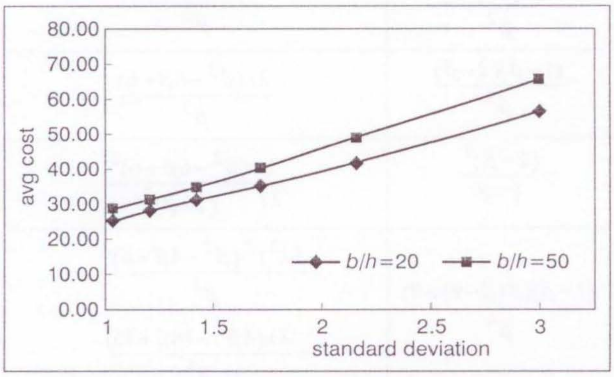

Figure 2 - Average cost vs. standard deviation of demand rate

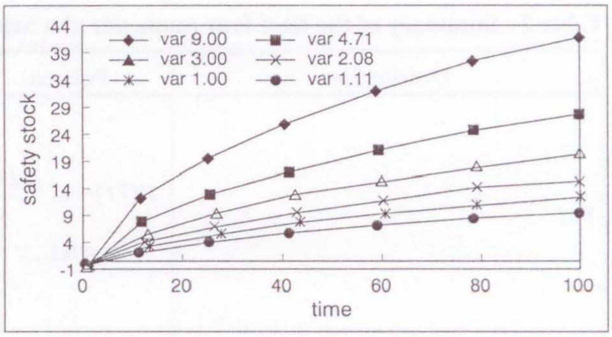

Figure 3 - Safety stock vs. time at different variances of demand rate, $b/h=20$

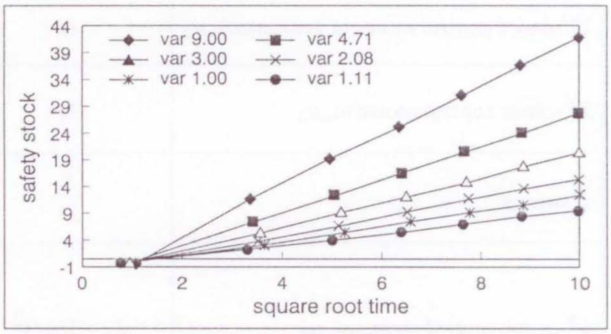

Figure 4 - Safety stock vs. square root of time at different variances of demand rate, $b/h=20$

Safety stock is defined to be the difference of the cumulative production immediately before the replenishment and the expected cumulative demand. Figures 3 and 4 illustrate the behaviour of stockout as a function of time. Again, we notice there is a strong linear relationship between safety stock and the square root of time. This is more clearly demonstrated in Table 4, where regression functions and their coefficients of determination R^2 are listed.

Table 4 - Linear regression of the curves in Figure 4

Variance of demand rate	Linear regression curve	R^2
9.00	$y = 4.6737x - 4.9309$	0.9977
4.71	$y = 3.0915x - 3.6661$	0.9997
3.00	$y = 2.3081x - 3.0559$	0.9999
2.08	$y = 1.7665x - 2.4953$	0.9986
1.50	$y = 1.4211x - 2.2408$	0.9985
1.11	$y = 1.0760x - 1,7370$	0.9968

We then use these regression curves to predicated safety stock at the same time for different variances of demand rate and analyse the impact from the variance. The results are shown in Figures 5 and 6. Apparently, the safety stock increases with the variance and it is more likely that it increases linearly with the standard deviation of the demand rate. Table 4 lists the outcome of regression.

Table 5 - Linear regression of the curves in Figure 6

Time t	Linear regression curve	R^2
80	$y = 14.841x - 7.9193$	0.9994
60	$y = 12.640x - 6.8862$	0.9994
40	$y = 10.029x - 5.6607$	0.9993
20	$y = 6.6261x - 4.0636$	0.9990

8. SUMMARY

The paper aims to study a compound distribution in production-inventory system. By applying the transform methods, the probability of demand can be described in terms of a mixture of the Laplace and z-transforms. Consequently, it is easy to derive its moments and other parameters for the distribution, such as variance, skewness, and kurtosis.

We have then investigated the possibility to extend our previous optimal production plan model to cover a stochastic compound demand process where the size of demand is Geometric distributed. The stockout function and optimisation equations are addressed in this case. For other compound demand process, we can basically follow the same methodology but the stockout function cannot be expressed in a compact

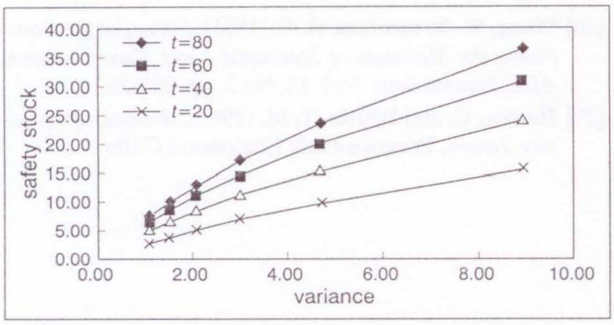

Figure 5 - Safety stock vs. variance of demand rate

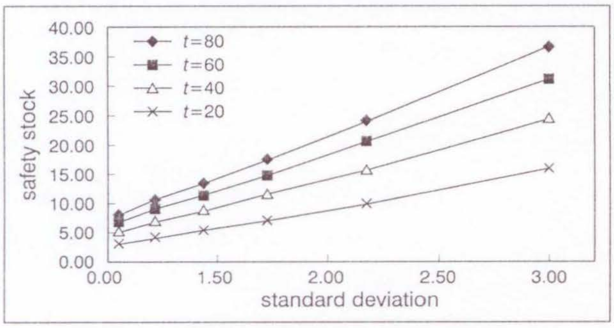

Figure 6 - Safety stock vs. standard deviation of demand rate

way in terms of Laplace transform and probably more computation is required for reaching the results.

Same as our previous study [18] numerical examples show again that the safety stock level is fairly linear with the square root of time. For our particular example in this study, it is also strongly illustrated that the safety stock level has a linear relationship with the standard deviation of demand rate.

For the future study, we would use this compound distribution for the situation when the lead time is a random variable in production-inventory system. High-order moments and their impact on optimal parameters in inventory control models, for instance (*s*, *S*) and (*s*, *Q*) models, are also of interest.

SAŽETAK

Složena distribucija je od interesa za ispitivanje problema zaliha, budući da daje fleksibilniji opis stohastičkih svojstava sustava u usporedbi s mnogim drugim pristupima kao što je postupak obnove. Međutim, zbog poteškoća u dobivanju analitičkih rezultata za složenu distribuciju, takav se tip ispitivanja obično ograničuje na traženje dobre aproksimacije koja bi zamijenila kompleksni model. Ovaj rad istražuje mogućnost proširenja postojećeg stohastičkog modela zalina na postupak složene potražnje. Metode transformacije ponovno imaju važnu ulogu u analizi radi određivanja stohastičkih svojstava složene distribucije.

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