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Science in Traffic
Original Scientific Paper
U. D. C.: 621.396.67:624.075.23:621.395.74
Accepted: Feb. 13, 2000
Approved: Sep. 13, 2000

STABILITY OF ANTENNA-COLUMN OF NEGLECTABLE OWN WEIGHT IN DESIGN OF WIRELESS TRAFFIC NETWORK

ABSTRACT

The paper deals with the problem of elastic stability of antenna-column of neglectable own weight subjected to the action of effective load on its top. The determination of critical buckling force is based upon the derived exact solution to the equilibrium equation of the column of continuously variable cross section loaded with constant compressive force.

KEY WORDS

antenna-column, elastic stability, critical buckling force

1. INTRODUCTION

The approximate design of antenna-columns by the 2nd order theory demands the knowledge of their critical buckling loads. Multiplication of bending moments and displacements determined following the 1st order theory by amplification factor

$$\alpha = 1 - \frac{F}{F_{cr}} \tag{1}$$

where F denotes the actual axial load of the column and F_{cr} denotes critical buckling load, produces bending moments and displacements by the 2nd order theory.

This paper deals with the elastic stability of antenna-column of neglectable own weight. In fact, the paper presents the study of elastic stability of a cantilevered column of continuously variable cross section subjected to the action of constant axial compressive force. The determination of the critical buckling force is based upon the exact solution to the equilibrium equation of the column. In the general case of axial loading the equilibrium equation is a homogeneous differential equation

$$\left[EI(x)w''\right]'' + F_a(x)w'' + q_a(x)w' = 0 \tag{2}$$

where E is the modulus of elasticity, $I(x)$ axial moment of inertia of the column cross section, $q_a(x)$ axial distributed load and

$$F_a = F + \int_0^x q_a(x)dx \tag{3}$$

is the total axial load in the cross section x. F is the constant axial compressive force acting in the cross section $x = 0$.

In the particular case of a column of neglectable own weight subjected to the action of constant axial compressive force, the equation (3) takes the form

$$\left[EI(x)w''\right]'' + Fw'' = 0 \tag{4}$$

as $q_a(x) = 0$. Here $(\ )' = \frac{d}{dx}(\ )$ and $(\ )'' = \frac{d^2}{dx^2}(\ )$.

2. THE SOLUTION TO THE EQUILIBRIUM EQUATION

Fig. 1 shows a cantilevered column of continuously variable cross section subjected to the action of the constant force F . The moment of inertia of the column cross section is varying along the column by the exponential law

$$I(x) = \left(\frac{x}{a}\right)^n I_1 \tag{5}$$

where n depends upon the geometry of the column and its cross section.

The two-times integration of (2) gives

$$EI(x)w'' + Fw = C_1x + C_2 \tag{6}$$

where C_1 and C_2 are constants of integration. Substitution of (5) in (6) produces

$$x^n w'' + k^2 w = K_1x + K_2 \tag{7}$$

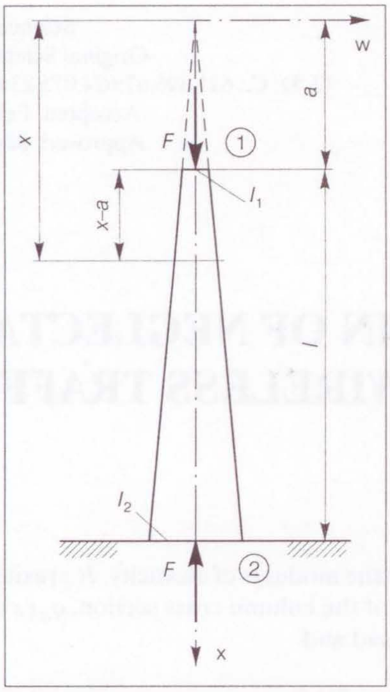

Figure 1 - Column of variable cross section subjected to the constant axial compressive force

where

$$k^2 = \frac{Fa^n}{EI_1} \tag{8}$$

and K_1 and K_2 are constants.

The homogeneous part of the equation (7) by substitution

$$t = x^{1-\frac{n}{2}} \tag{9}$$

can be transformed into

$$t^2 \frac{d^2 w}{dt^2} + \frac{n}{n-2} \cdot t \cdot \frac{dw}{dt} + \frac{4k^2}{(n-2)^2} \cdot t^2 w = 0 \tag{10}$$

By the power series method (the Frobenius method) the solution to the equation (10) can be sought in the form

$$w = t^r (a_0 + a_1 t + a_2 t^2 + \ldots) \tag{11}$$

The equation (10) can be written in the form

$$t^2 w'' + tb(t) w' + c(t) w = 0 \tag{12}$$

The indicial equation of the differential equation (12) is

$$r(r-1) + b(0) r + c(0) = 0 \tag{13}$$

The exponent r in (11) takes the values of the roots r_1 and r_2 of the indicial equation (13). The power series w_1 and w_2 thus obtained after rearranging get the form of power series by which Bessel's functions of the first kind of orders $(2-n)^{-1}$ and $-(2-n)^{-1}$ are defined. Finally, the solution to the homogeneous part of the equation (7) is obtained in the form

$$w_h = A_1 w_1 + A_2 w_2 = A_1 \sqrt{x} \cdot J_{\frac{1}{2-n}}\left(\frac{2k}{n-2} \cdot x^{1-\frac{n}{2}}\right) +$$
$$+ A_2 \sqrt{x} \cdot J_{-\frac{1}{2-n}}\left(\frac{2k}{n-2} \cdot x^{1-\frac{n}{2}}\right) \tag{14}$$

where A_1 and A_2 are constants and $J_{\frac{1}{2-n}}$ and $J_{-\frac{1}{2-n}}$ are the above mentioned Bessel's functions. If $\frac{1}{2-n}$ takes the value of an integer m, the Bessel's functions $J_m(u)$ and $J_{-m}(u)$ are linearly dependent, because $J_{-m}(u) = (-1)^m J_m(u)$, $m = 1, 2, \ldots$, and thus there would not be two linearly independent solutions w_1 and w_2 in (14). In this case $J_{-m}(u)$ can be substituted by Bessel's function of the second kind (Neumann's function) of order m $N_m(u)$ which follows the same pattern as $J_m(u)$ in derivating and thus it can also be a solution to the homogeneous part of the equation (7). Therefore, for $n = 1$ and $n = 3$ it follows

$$(w_h)_{n=1} = B_1 \sqrt{x} \cdot J_1\left(2k\sqrt{x}\right) + B_2 \sqrt{x} \cdot N_1\left(2k\sqrt{x}\right) \tag{15}$$

$$(w_h)_{n=3} = B_1 \sqrt{x} \cdot J_1\left(\frac{2k}{\sqrt{x}}\right) + B_2 \sqrt{x} \cdot N_1\left(\frac{2k}{\sqrt{x}}\right) \tag{16}$$

where B_1 and B_2 are constants.

For $n = 4$ Bessel's functions of the first kind of orders 1/2 and -1/2 are obtained. These functions are defined as follows

$$J_{\frac{1}{2}}(u) = \sqrt{\frac{2}{\pi \cdot u}} \sin(u) \quad J_{-\frac{1}{2}}(u) = \sqrt{\frac{2}{\pi \cdot u}} \cos(u) \tag{17}$$

and hence

$$(w_h)_{n=4} = B_1 x \sin\left(\frac{k}{x}\right) + B_2 x \cos\left(\frac{k}{x}\right) \tag{18}$$

For $n = 2$ the solution (14) is not possible and therefore the solution to the homogeneous part of the equation (7) has to be determined in some other way. By introducing the substitution

$$x = ae^t$$

the homogeneous part of the equation (7) transforms into

$$\frac{d^2 w}{dt^2} - \frac{dw}{dt} + k^2 w = 0 \tag{19}$$

This is a homogeneous linear differential equation with constant coefficients with the corresponding characteristic equation

$$r^2 - r + k^2 = 0 \tag{20}$$

The roots r_1 and r_2 of the equation (20), considering substitution

$$k^2 = \frac{Fa^2}{EI_1} = b^2 + \frac{1}{4} \tag{21}$$

with the assumption

$$\frac{Fa^2}{EI_1} > \frac{1}{4} \tag{22}$$

determine the solution

$$(w_h)_{n=2} = B_1\sqrt{x}\sin\left(b\cdot\ln\frac{x}{a}\right) + B_2\sqrt{x}\cos\left(b\cdot\ln\frac{x}{a}\right) \tag{23}$$

The solution to the nonhomogeneous differential equation (7) can be obtained by the method of the variation of parameters. The equation (7) can be written in the form

$$w'' + \frac{k^2}{x^n}w = \frac{K_1x+K_2}{x^n} \tag{24}$$

The solution to (24) will take the form

$$w = C_1w_1(x) + C_2w_2(x) \tag{25}$$

where $w_1(x)$ and $w_2(x)$ are the two linearly independent solutions to the homogeneous part of (24). The method gives the system of equations

$$C_1'w_1 + C_2'w_2 = 0 \tag{26}$$

$$C_1'w_1' + C_2'w_2' = \frac{K_1x+K_2}{x^n} \tag{27}$$

with the solution

$$C_1' = -\frac{K_1x+K_2}{x^n}\cdot\frac{w_2}{W} \tag{28}$$

$$C_2' = \frac{K_1x+K_2}{x^n}\cdot\frac{w_2}{W} \tag{29}$$

where $W = w_1w_2' - w_1'w_2$ is the Wronskian of w_1, w_2. By the Liouville's formula

$$W(x) = W(x_0)e^{-\int_{x_0}^{x}a_1(x)dx} = W(x_0)e^0 =$$

$$= W(x_0) = W_0 \tag{30}$$

Here $a_1(x)$ is the coefficient multiplying w' on the left side of equation (24). It is obvious that $a_1(x) = 0$, and thus Wronskian W equals some constant W_0. From (28) it follows, considering (30)

$$C_1 = -\frac{1}{W_0}\int_{x_0}^{x}\frac{K_1x+K_2}{x^n}w_2(x)dx \tag{31}$$

Further, considering substitutions

$$p = -\frac{1}{n-2} \tag{32}$$

$$u = \frac{2k}{n-2}x^{1-\frac{n}{2}} \tag{33}$$

and hence

$$\frac{d}{dx} = \frac{d}{du}\cdot\frac{du}{dx} = -kx^{-\frac{n}{2}}\frac{d}{du} \tag{34}$$

it follows

$$w_2' = kx^{\frac{1-n}{2}}J_{p+1}(u) \tag{35}$$

and

$$w_2'' = -k^2\sqrt{x}\cdot x^{-n}J_p(u) = -\frac{k^2}{x^n}w_2(x) \tag{36}$$

Substitution of (36) in (31) gives

$$C_1 = \frac{1}{W_0k^2}\int_{x_0}^{x}(K_1x+K_2)w_2''(x)dx \tag{37}$$

By adding $K_1w_2'(x) - K_1w_2'(x)$ to (K_1x+K_2) in (37), C_1 becomes

$$C_1 = \frac{1}{W_0k^2}\int_{x_0}^{x}\{[K_1w_2'(x) + (K_1x+K_2)w_2''(x)] -$$

$$-K_1w_2'(x)\}dx =$$

$$= \frac{1}{W_0k^2}\int_{x_0}^{x}\left\{[(K_1x+K_2)w_2'(x)]' - K_1w_2'(x)\right\}dx =$$

$$= \frac{1}{W_0k^2}[(K_1x+K_2)w_2'(x) - K_1w_2(x)] - \overline{A}_1 \tag{38}$$

where $\overline{A}_1$ is the constant following for $x = x_0$. Analogously it follows

$$C_2 = \frac{1}{W_0k^2}[(K_1x+K_2)w_1'(x) - K_2w_1(x)] - \overline{A}_2 \tag{39}$$

Considering (38) and (39) the general solution to (24) takes the form

$$w = C_1w_1(x) + C_2w_2(x) = \overline{A}_1w_1(x) + \overline{A}_2w_2(x) +$$

$$+\frac{K_1x+K_2}{k^2}\cdot\frac{w_1(x)w_2'(x) - w_1'(x)w_2(x)}{W_0} \tag{40}$$

If $\overline{A}_1$ and $\overline{A}_2$ are denoted $\overline{A}_1 = A_1$, $\overline{A}_2 = A_2$ (arbitrary constants), and considering that by (30) Wronskian $W(x) = w_1(x)w_2'(x) - w_1'(x)w_2(x) = W_0$, it follows

$$w = A_1w_1(x) + A_2w_2(x) + \frac{K_1x+K_2}{k^2} =$$

$$= w_h + \frac{K_1x+K_2}{k^2} \tag{41}$$

3. DETERMINATION OF CRITICAL BUCKLING FORCE

According to Fig. 2. it follows

$$F_T = F_N\sin\alpha + F_Q\cos\alpha \tag{42}$$

$$F_L = F_N\cos\alpha - F_Q\sin\alpha \tag{43}$$

and hence

$$F_T = F_L\,\mathrm{tg}\,\alpha + F_Q\left(\frac{\sin^2\alpha}{\cos\alpha} + \cos\alpha\right) \tag{44}$$

For a very small α, $\cos\alpha \approx 1$, $\mathrm{tg}\,\alpha \approx \alpha = w'$, $\sin^2\alpha \approx \mathrm{tg}^2\alpha \approx \alpha^2 \approx 0$,

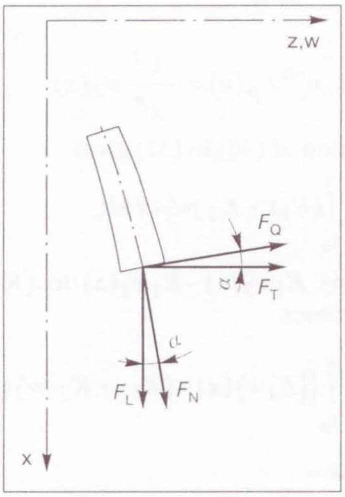

Figure 2 - Normal, shearing, longitudinal and transverse force

and hence

$$F_T = F_L w' + F_Q \tag{45}$$

where longitudinal force

$$F_L = -F_a = -F \tag{46}$$

and shearing force

$$F_Q = \frac{dM}{dx} = \frac{d}{dx}[-EI(x)w''] = -[EI(x)w'']' \tag{47}$$

According to (5) and (8)

$$F = k^2 E \frac{I_1}{a^n} = k^2 E \frac{I(x)}{x^n} \tag{48}$$

Substitution of (47) and (48) in (45), considering (5), gives

$$\begin{aligned} F_T &= -[EI(x)w'']' - EI(x)\frac{k^2}{x^n}w' = \\ &= -EI(x)w''' - E\frac{dI(x)}{dx}w'' - EI(x)\frac{k^2}{x^n}w' = \\ &= -EI(x)w''' - EI_1\frac{nx^{n-1}}{a^n}w'' - EI(x)\frac{k^2}{x^n}w' = \end{aligned} \tag{49}$$

The boundary conditions for the column are the following :

1) for $x = a + l$, $w = 0$ (50)

1) for $x = a + l$, $w' = 0$ (51)

1) for $x = a$, $M = 0$ (52)

1) for $x = a$, $F_T = 0$ (53)

The boundary conditions (50) and (51) expressed by (41) yield

$$A_1 w_1(a+l) + A_2 w_2(a+l) + K_1 \frac{a+l}{k^2} + K_2 = 0 \tag{54}$$

$$A_1 w_1'(a+l) + A_2 w_2'(a+l) + K_1 \frac{1}{k^2} = 0 \tag{55}$$

From (52) it follows

$$M(x=a) = -EI(a)w''(a) = 0$$

and hence

$$w''(a) = 0 \tag{56}$$

which gives

$$A_1 w_1''(a) + A_2 w_2''(a) = 0 \tag{57}$$

From (53) and (49), considering (56), it follows

$$-EI(a)w'''(a) - EI_1 \frac{na^{n-1}}{a^n} w''(a) - EI(a)\frac{k^2}{a^n} w'(a) = 0$$

$$w'''(a) + \frac{k^2}{a^n} w'(a) = 0 \tag{58}$$

and hence

$$A_1\left[w_1'''(a) + \frac{k^2}{a^n} w_1'(a)\right] + \\ + A_2\left[w_2'''(a) + \frac{k^2}{a^n} w_2'(a)\right] + \frac{K_1}{a^n} = 0 \tag{59}$$

The equations (54), (55), (57) and (59) form a system of four homogenous linear equations with constants A_1, A_2, K_1 and K_2 as unknowns.

The condition of a non-trivial solution to the system demands that its determinant equals zero. As the terms of the determinant D are functions of k^2, i.e., by (8) functions of F, any value of F making the determinant equal to zero is a force that causes buckling of the column. The critical force F_{cr} is the smallest value among them.

The equations $D(k) = 0$ and the expressions defining F_{cr} for particular exponents n, considering expressions following from (5) and (8)

$$m = \frac{I_2}{I_1} = \left(\frac{a+l}{a}\right)^n \tag{60}$$

$$k = \frac{F(a+l)^n}{EI_2} \tag{61}$$

take the forms :

n = 1 $\quad J_1(t)N_0\left(\sqrt{m}\cdot t\right) - N_1(t)J_0\left(\sqrt{m}\cdot t\right) = 0 \quad (62)$

$$t = 2k\sqrt{a} \tag{63}$$

$$F_{cr} = \frac{(m-1)^2}{m} \cdot \frac{t_0^2}{4} \cdot \frac{EI_2}{l^2} \tag{64}$$

n = 2 $\quad \mathrm{tg}(t) + \frac{4}{\ln m} t = 0 \quad (65)$

$$t = b \cdot \ln\sqrt{m}, \; b = \sqrt{k^2 - \frac{1}{4}} \tag{66}$$

$$F_{cr} = \frac{\left(\sqrt{m}-1\right)^2}{m}\left[\left(\frac{2t_0}{\ln m}\right)^2 + \frac{1}{4}\right] \cdot \frac{EI_2}{l^2} \tag{67}$$

n = 3 $\quad J_1\left(\sqrt[6]{m}\cdot t\right)\cdot\left[N_1(t) - \frac{t}{2}N_0(t)\right] - \\ \quad - N_1\left(\sqrt[6]{m}\cdot t\right)\cdot\left[J_1(t) - \frac{t}{2}J_0(t)\right] = 0 \quad (68)$

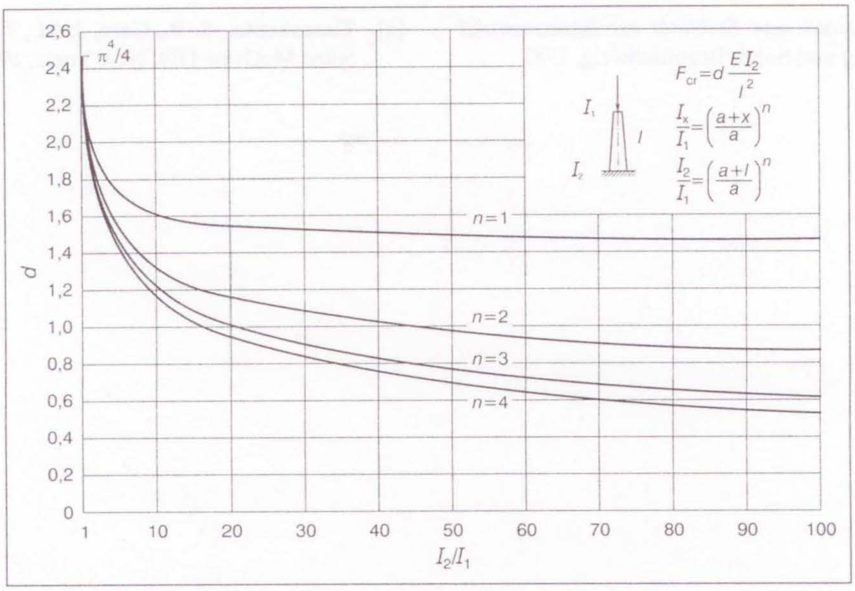

Diagram 1

$$t=\frac{2k}{\sqrt{a+l}} \tag{69}$$

$$F_{cr}=\left(\frac{\sqrt[3]{m}-1}{\sqrt[3]{m}}\right)^2\cdot\frac{t_0^2}{4}\cdot\frac{EI_2}{l^2} \tag{70}$$

n = 4 $$\operatorname{tg} t-\frac{t}{\sqrt[4]{m}-1}=0 \tag{71}$$

$$t=\frac{k}{a}-\frac{k}{a+l} \tag{72}$$

$$F_{cr}=\frac{1}{\sqrt{m}}t_0^2\frac{EI_2}{l^2} \tag{73}$$

The quantity t_0 appearing in (64), (67), (70) and (73) denotes the smallest root of the equations (62), (65), (68)and (71) respectively.

4. RESULTS

The values of F_{cr} calculated from the equations stated above are given in Diagram 1. Diagram 1 shows the dependence of dimensionless factor

$$d=\frac{F_{cr}l^2}{EI_2} \tag{74}$$

upon the ratio I_2 / I_1 of the moments of inertia of the cross sections at the ends of the column and upon the exponent n of the parabola describing the varying of the moment of inertia of the cross section along the column.

The exponents $n = 1$, $n = 2$, $n = 3$ and $n = 4$ describe the geometries of the columns which are most frequently used in practice.

Exponent $n = 1$ refers to the column of a rectangular cross section of constant height h and width $b_x(b_x > h)$ linearly dependent upon x.

Exponent $n = 2$ approximately refers to a column constructed from 4 rods laid along the edges of virtual truncated pyramid connected with the lattice fillment of neglectable weight.

Exponent $n = 3$ on the one hand refers to the column of rectangular cross section with constant width b and height $h_x(h_x < b)$ linearly dependent upon x and on the other hand it approximately refers to the column in the form of a conical tube of constant thickness of the wall.

Exponent $n = 4$ refers to the columns in the form of truncated cone or pyramid and hollow truncated cone or pyramid.

SAŽETAK

STABILNOST ANTENSKIH STUPOVA ZANEMARIVE VLASTITE TEŽINE U PROJEKTIRANJU MREŽE BEŽIČNOG PROMETA

Članak se bavi problemom elastične stabilnosti antenskog stupa zanemarive vlastite težine izloženog djelovanju korisnog tereta na njegovu vrhu. Određivanje kritične sile izvijanja temelji se na dobivenom egzaktnom rješenju jednadžbe ravnoteže stupa kontinuirano promjenjljivog presjeka opterećenog konstantnom tlačnom silom.

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