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ON THE CAR-FOLLOWING THEORY

SUMMARY

The article deals with bumper-to-bumper driving. A systemic approach to car-following (bumper-to-bumper driving) is described and the mathematical model has been developed and solved. With this approach an algorithm for analytical and numerical solutions has been developed.

1. INTRODUCTION

The current problem of bumper-to-bumper driving has been dealt with in a number of ways. All the models used for the simulation and control of bumper-to-bumper traffic are derived from physical laws of motion. Differential equations describing this motion depend on different approaches to the problem.

The system of differential equations for our problem can be obtained from the requirement [1] that the velocity error $\varepsilon_{1k}(t)=v_k(t)-v_{k+1}(t)$ and the spacing error $\varepsilon_{2k}(t)=x_k(t)-x_{k+1}(t)-S$ are minimal. If this condition is fulfilled, we get

$$\dot{\varepsilon}_{2k}(t)=\varepsilon_{1k}(t)-\varepsilon_{1k+1}(t)$$
$$\dot{\varepsilon}_{1k}(t)=-\frac{\sigma_k}{M_k}\varepsilon_{1k}(t)+\frac{1}{M_k}\varepsilon_{3k}(t) \tag{1}$$

In (1) $\varepsilon_{3k}(t)=F_k(t)-F_{vk}$ and F_{vk} is the steady state tractive effort required to propel the vehicle k at system speed. It is necessary to define the most advantageous manner of system regulation. On the other hand an entire family of the car-following models has been investigated based on the following general form [7]:

$$\ddot{x}_{k+1}(t+T)=c\dot{x}_{k+1}^{m}(t+T)\frac{\left[\dot{x}_k(t)-\dot{x}_{k+1}(t)\right]}{\left[x_k(t)-x_{k+1}(t)\right]^{l}} \tag{2}$$

In this article the problem of the car-following theory will be presented with another supposition.

2. SETTING THE PROBLEM

Consider the line of n+1 $(n\in\mathcal{N}, n>1)$ vehicles. The first vehicle is moving at a velocity v_0 ($v_0=v_0(t)>0$), and the other n vehicles are following, adjusting their velocity to the vehicle in front of them.

The following notation will be used in the analysis (see Fig. 1):

$n+1$ – the number of vehicles in the line of traffic,

k – an index number, $k=0,1,2,...,n$,

$x_k=x_k(t)$ – the coordinate of the k-th vehicle's front

$v_k=v_k(t)$ – the velocity of the k-th vehicle,

$v_0=v_0(t)$ – the velocity of the leading vehicle,

$D_k=D_k(t)$ – the postulated legal distance of separation of the (k-1)-th and k-th vehicles.

It follows from Fig. 1 that:

$$D_k(t)=x_{k-1}(t)-x_k(t) \tag{3}$$

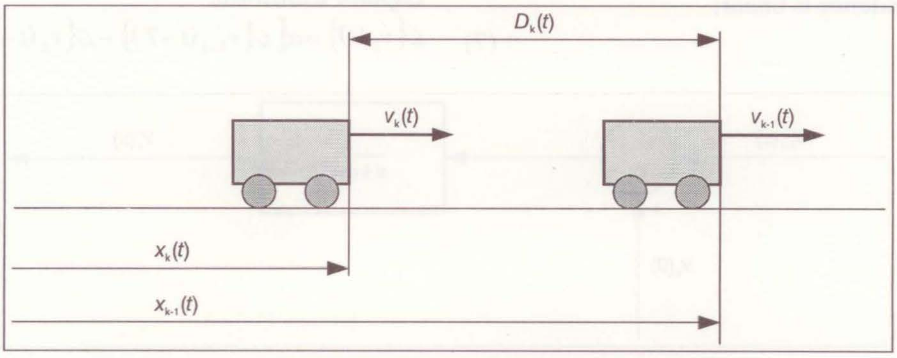

Figure 1 - Relations between vehicles in the line

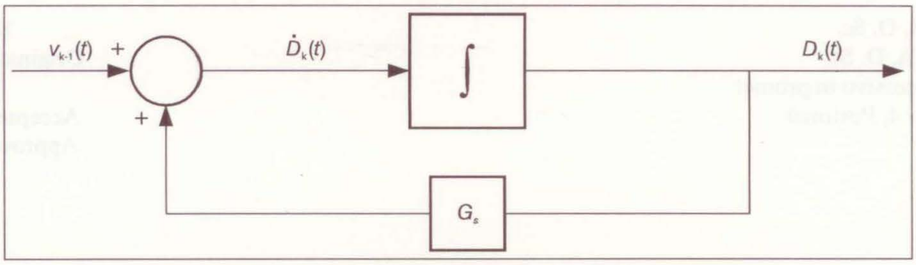

Figure 2 - Simulation diagram of the vehicle-driver system

and, because $v_{k-1}(t)=\frac{dx_{k-1}(t)}{dt}$ and $v_k(t)=\frac{dx_k(t)}{dt}$, it follows (3)

$$\frac{dD_k(t)}{dt}=v_{k-1}(t)-v_k(t) \tag{4}$$

Equation (4) satisfies the physical requirements of the problem.

Let us suppose that we can obtain dynamic equations governing the line of traffic by insisting that each vehicle keeps the required legal distance [5]. Each driver pays attention to the vehicle in front of him, which means that

$$v_k=v_k\left(D_k\right),\quad k=1,2,\ldots,n \tag{5}$$

With this assumption, equation (5) can be written in the following way:

$$\dot{D}_k+v_k\left(D_k\right)=v_{k-1},\quad k=1,2,\ldots,n \tag{6}$$

If we consider each vehicle as dynamic linear system, differential equations (6) describe the dynamics of the vehicle in the line. Equations (6) are dynamic equations of the system of vehicles and represent the mathematical model of a continuous dynamic control system. The response of the system to this input function is calculated, i. e. the distance $D_k(t)$, which k-th driver ($k = 1,2,\ldots,n$) with optimum control of his vehicle (system) adjusts to the motion of the vehicle in front of him. This distance is the time function and represents the law which drivers in a line of vehicles abide by, i.e. the law of the line of vehicles. Simulation diagram of this mathematical model is given in Fig. 2.

G_s is the feed-back operator describing the function's dependence (5).

The simplest type of motion to analyse is the one in which the dependence is linear:

$$v_k=\alpha_k D_k \tag{7}$$

Relations (7) assume that every driver in the line adjusts the velocity of his vehicle to the velocity of the vehicle in front of him in the linear manner. The coefficient $\tau_k=\alpha_k^{-1}$ represents the total time in which the k-th driver would cover the distance D_k to the vehicle in front of him if the latter suddenly stopped. This time is made up of two parts: the driver's reaction time T_k and the time ζ_k which is needed for displacement D_k in the physical sense:

$$\tau_k=T_k+\zeta_k \tag{8}$$

Without affecting the generality of the model, we may simplify it and accept that on average all these times are the same and therefore

$$\tau_1=\tau_2=\ldots=\tau_n=\tau \tag{9}$$

$$T_1=T_2=\ldots=T_n=T \tag{10}$$

$$\zeta_1=\zeta_2=\ldots=\zeta_n=\zeta \tag{11}$$

$$\alpha_1=\alpha_2=\ldots=\alpha_n=\alpha \tag{12}$$

By inserting (9) - (12) into the equations of the system (6) we get a mathematical model in the form of

$$\dot{v}_k(t)=\alpha\left[v_{k-1}(t-T)-v_k(t-T)\right],\quad k=1,2,\ldots,n \tag{13}$$

Equations (13) are dynamic equations of the system of vehicles and are non-homogeneous linear equations of the 1st order. The right side of (8) is the input of the dynamic driver-vehicle system. We will calculate the output of the system, i.e. velocity $v_k(t)$ of the k-th driver. The leading driver's velocity v_0 is known. It is arbitrary and depends on the weather conditions and the state of the road surface.

$$D_k(0)={}^{(0)}D_k,\quad v_k(0)={}^{(0)}v_k \tag{14}$$

We can solve the system of equations (13) by using Laplace transform:

$$\mathcal{L}\left\{\dot{v}_k(t)\right\}=\alpha\left[\mathcal{L}\left\{v_{k-1}(t-T)\right\}-\mathcal{L}\left\{v_k(t-T)\right\}\right] \tag{15}$$

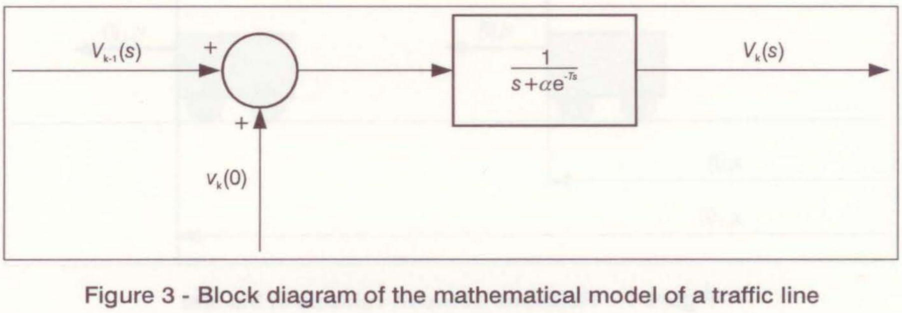

Figure 3 - Block diagram of the mathematical model of a traffic line

We denote it $\mathcal{L}\{v_k(t)\} = V_k(s)$ and obtain (15)

$$V_k(s) = \frac{\alpha e^{-Ts} V_{k-1}(s)}{s+\alpha e^{-Ts}} + \frac{v_k(0)}{s+\alpha e^{-Ts}} \quad k=1,2,\dots,n \tag{16}$$

The block diagram of equation (16) is shown in Fig. 3.

To obtain the solution in real time-space, equation (16) cannot be used with inverse Laplace transform, so it must be rearranged and stated in a different form.

For $k = 1$ we first obtain

$$V_1(s) = \frac{\alpha e^{-Ts} V_0(s) + v_1(0)}{s+\alpha e^{-Ts}}$$

For $k = 2$ we obtain

$$V_2(s) = \frac{\alpha e^{-Ts} V_1(s) + v_2(0)}{s+\alpha e^{-Ts}} =$$

$$= \frac{\alpha^2 e^{-2Ts} V_0(s)}{\left(s+\alpha e^{-Ts}\right)^2} + \frac{\alpha e^{-Ts} v_1(0)}{\left(s+\alpha e^{-Ts}\right)^2} + \frac{v_2(0)}{\left(s+\alpha e^{-Ts}\right)}$$

Generally, we obtain

$$V_k(s) = \frac{\alpha^k e^{-kTs} V_0(s)}{\left(s+\alpha e^{-Ts}\right)^k} + \sum_{r=0}^{k-1} \frac{\alpha^r e^{-rTs} v_{k-r}(0)}{\left(s+\alpha e^{-Ts}\right)^{r+1}},$$

$$k=1,2,\dots,n \tag{17}$$

Because ([2], [3]) the series are convergent

$$\frac{\alpha^k e^{-kTs}}{\left(s+\alpha e^{-Ts}\right)^k} = \sum_{r=k}^{\infty} (-1)^{r-k} \frac{(r-1)!\alpha^r e^{-rTs}}{(k-1)!(r-k)!s^r},$$

$$\left|\frac{\alpha e^{-Ts}}{s}\right| < 1 \tag{18}$$

$$\frac{\alpha^k e^{-kTs}}{\left(s+\alpha e^{-Ts}\right)^{k+1}} = \sum_{r=k}^{\infty} (-1)^{r-k} \frac{r!\alpha^r e^{-rTs}}{k!(r-k)!s^r},$$

$$\left|\frac{\alpha e^{-Ts}}{s}\right| < 1 \tag{19}$$

it follows from (17)

$$V_k(s) = \sum_{r=k}^{\infty}\left[(-1)^{r-k} \frac{(r-1)!V_0(s)\alpha^r e^{-rTs}}{(k-1)!(r-k)!s^r}\right] +$$

$$+\sum_{r=0}^{k-1}\left[\frac{v_{n-r}(0)}{r!} \sum_{l=r}^{\infty} (-1)^{l-r} \frac{l!\alpha^l e^{-lTs}}{(l-r)!s^{l+1}}\right] \tag{20}$$

Equation (20) undergoes inverse Laplace transformation and so we obtain the solution of a differential equation (17), i.e. the speed of the k-th driver (k = 1,2,3,...,n) in the traffic line:

$$v_k(t) = \sum_{r=k}^{\infty}\left[\frac{(-1)^{r-k}\alpha^r}{(k-1)!(r-k)!}\mathcal{F}_r(t-rT)\right] +$$

$$+\sum_{r=0}^{k-1}\left[\frac{v_{n-r}(0)}{r!} \sum_{m=r}^{\infty} \frac{(-1)^{m-r}\alpha^m e^{-mTs}}{(m-r)!}(t-mT)_+^m\right] \tag{21}$$

Here

$$\mathcal{F}_r(t) = \mathcal{L}^{-1}\left\{V_0(s)\frac{(r-1)!}{s^r}\right\} =$$

$$= \begin{cases} \int_0^t v_0(t-\xi)\xi^{k-1}d\xi & t \ge 0 \\ 0 & t<0 \end{cases} \tag{22}$$

$$(t-mT)_+ = \begin{cases} t-mT & t \ge mT \\ 0 & t > mT \end{cases} \tag{23}$$

To complete the solution we have to estimate its asymptotic behaviour. Since $\lim_{t\to\infty} V_k(t) = \lim_{s\to 0} sV_k(s)$ it follows from (17)

$$\lim_{t\to\infty} V_k(t) = \lim_{s\to 0} sV_0(s) \tag{24}$$

3. SPECIAL EXAMPLES

1. Drivers react without delay: $T = 0$.
2. The velocity of the leading vehicle differs in time: $v_0(t) = v_0 + a_0 t$.
3. The combination of both conditions: $T = 0$ and $v_0(t) = v_0 + a_0 t$.

When $T = 0$ we obtain the model from (13) in the form of

$$\dot{v}_k(t) = \alpha\left[v_{k-1}(t) - v_k(t)\right], \quad k=1,2,\dots,n \tag{25}$$

Hence

$$V_k(s) = \frac{\alpha V_{k-1}(s)}{s+\alpha} + \frac{v_k(0)}{s+\alpha}$$

and

$$v_k(t) = \alpha e^{-\alpha t}\int_0^t e^{\alpha\xi} v_{k-1}(\xi)d\xi + v_k(0)e^{-\alpha t} = \tag{26}$$

$$= e^{-\alpha t}\left[\alpha\int_0^t e^{\alpha\xi} v_{k-1}(\xi)d\xi + {}^{(0)}D_k\right]$$

Let us state the leading vehicle's velocity as $v_0(t) = v_0 + a_0 t$. The leading vehicle is accelerating when $a_0 > 0$, decelerating when $a_0 < 0$ and moving at constant velocity when $a_0 = 0$. In this case equation (21) is stated in the form of

$$v_k(t) = \sum_{r=k}^{\infty}\left[\frac{(-1)^{r-k}\alpha^r}{(k-1)!(r-k)!}\int_0^t \left(v_0 + a_0(t-rT-\xi)\right)\xi^{k-1}d\xi\right] +$$

$$+\sum_{r=0}^{k-1}\left[\frac{v_{n-r}(0)}{r!} \sum_{m=r}^{\infty} \frac{(-1)^{m-r}\alpha^m e^{-mTs}}{(m-r)!}(t-mT)_+^m\right] \tag{27}$$

In the case $(T=0) \wedge \left(v_0(t) = v_0 + a_0 t\right)$ we obtain

$$v_k(t) = u + \left[v_0 - u + a_0(t-k\tau)\right]\left(1 - e^{-\frac{t}{\tau}}\sum_{i=0}^{k-1}\frac{1}{i!}\left(\frac{t}{\tau}\right)^i\right) +$$

$$+a_0\frac{\tau e^{-\frac{1}{\tau}}}{(k-1)!}\left(\frac{t}{\tau}\right)^k \tag{28}$$

$$u = \frac{D_0}{\tau} \tag{29}$$

The initial velocity of the k-th vehicle follows from (28):

$$v_k(0) = u + \left[v_0 - u - a_0 k\tau\right](1-1) + 0 = u \tag{30}$$

Expression (29) presents the initial velocity of a line of vehicles, i.e. the velocity at time $t=0$.

4. NUMERICAL EXAMPLES

Computer simulation with "Maple V" software tools was done for some special examples. The graphs of corresponding response functions are given in Fig. 4 - 6.

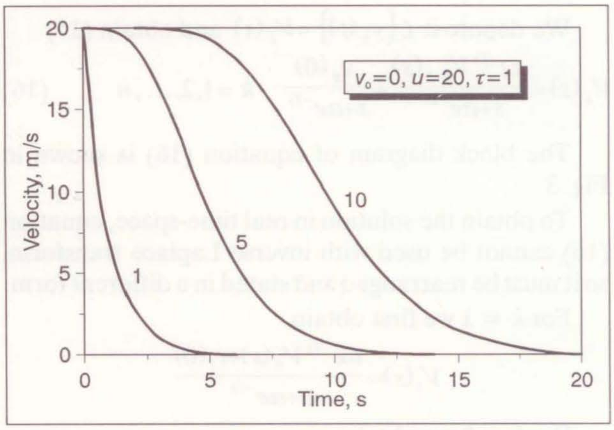

Figure 4 - $v_0=0$, $u=20$ m s^{-1}, $\tau=1$ s , $a_0=0$

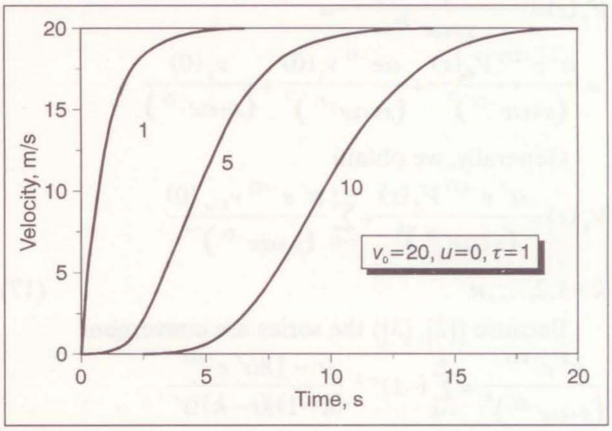

Figure 5 - $v_0=20$ m s^{-1}, $u=0$, $\tau=1$ s , $a_0=0$

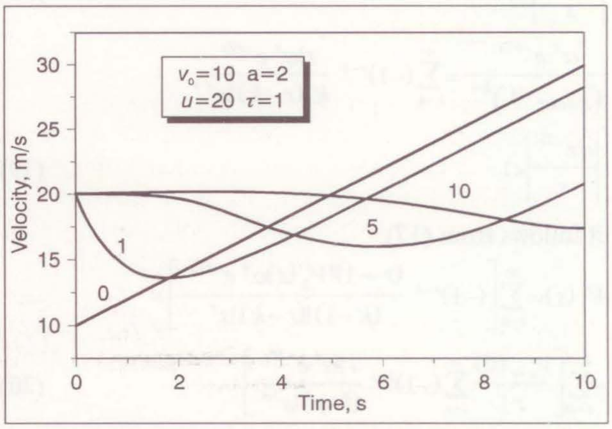

Figure 6 - $v_0=10$ m s^{-1}, $u=20$ m s^{-1}, $\tau=1$ s , $a_0=2$ m s^{-2}

5. CONCLUSION

The mathematical model of vehicle behaviour in a line of vehicles must be based on the real laws of physics. Solving differential equations which represent the mathematical model of a continuous dynamic system requires new approaches and the use of computers. Algorithms obtained in this way are good foundation for computer simulations which are an important tool for the study of road capacities.

POVZETEK

VOŽNJA VOZIL V KOLONI

V prispevku obravnavamo vožnjo vozil v koloni. Opisan je sistemski pristop k temu problemu, kreiran in rešen je relevanten matematični model. S takšnim pristopom uspemo razviti algoritem za analitične in numerične rešitve danega problema.

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