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SELECTION OF OPTIMAL TRAFFIC SIGNAL CYCLE USING GRAPHS

ABSTRACT

The paper deals with the traffic management of a traffic signal-controlled intersection. One set of movements consists of all the possible streams at an intersection that occur simultaneously. The set of movements changes cyclically during the control cycle. Within a cycle, each stream has to get at least once free passage, effective green time. Each set has to define the traffic flow at the intersection with no crossing points (points of conflict). In the cyclical order of the set of movements, the intersection capacity is greater if the number of sets within the cycle is lower, and "overlapping" between the sets greater. Using graph theory, two intersections in the city of Zagreb have been analysed: Dubrovnik Avenue – Večeslava Holjevca Avenue (61 traffic accidents a year) and Savska Road – Street of the City of Vukovar (36 traffic accidents a year).

KEY WORDS

traffic, theory of graphs, optimisation

1. INTRODUCTION

All the more complex traffic problems today are most frequently solved by mathematical methods. Mathematical modelling in traffic is used both for solving everyday problems in practice and in scientific research. Mathematical models and optimisation methods in traffic systems represent the implementation of quantitative methods for solving complex problems in the system of people, transport vehicles, traffic routes, cargo, traffic objects – devices, information and capital. The task of these methods is to provide the decision makers with the possibility of setting technical solutions and organisational decisions on scientific bases and to make them possibly rely on measurable values.

The application of graph theory over the past ten years has been very intensive in solving engineering traffic problems. Many scientific methods of optimising technical solutions and traffic organisation rely on the theory of graphs.

Graph theory, as a special mathematical discipline, is only about five decades old. Over the past twenty years, the theory of graphs has experienced an extraordinary intensive development owing to the increasing production and application of computers.

In this way, graph has developed from the auxiliary diagram used to illustrate various problems into the object of comprehensive mathematical theory. The mathematical apparatus of graph theory allows numerous problems with finite sets, from very diverse scientific fields, to be formulated and solved in a unique way. The application of the graph theory and its methods occupy today a significant place in the traffic theory, electrical circuit theory, in the theory of automatic control systems, theory of finite automatons, operations research, theory of reliable transmission of information, economic sciences, chemistry, sociology, etc. The main reason for such a wide scope of applications lies in the first place in the clear presentation provided by a graph, which is close to intuitive presentation.

Generally speaking, graph is a family of dots or circles, called vertices or nodes, together with lines connecting nodes, that are called arcs, branches or edges. Graphs can be found in many scientific fields and other human activities. Graphs can be used to represent networks of routes, railway lines, telephone and other telecommunication networks, air routes, document flows, etc. Graphs appear in chemistry as molecule structure formulas, as computer program diagrams, in the science of work organisation as network diagrams that describe complex projects, in sociology they are used to represent relations in a certain group of people, etc.

The first scientific work in the graph theory was Euler's work in 1936. Over the past, important contributions to graph theory were provided by G.

Kirchhoff, C. Kuratowski and D. König. In Croatia works on graph theory were published by: Đ. Kurepa, V. Dévidé, D. Blanuša, V. Niče, V. Sedmak, D. Veljan, H. Pašagić.

The increase in safety at intersections was largely affected by adequate and well established traffic signalling (traffic lights) with good organisation of vehicle flows within a cycle (traffic signal phases) by eliminating crossing paths of vehicles that drive through the intersection (conflict points) and by maximal efficiency of the intersection within a cycle.

In this paper the intersections in Zagreb have been analysed at which a great number of traffic accidents occur, and that are very complex due to the presence of tramways. Based on this criterion, the intersection of the Dubrovnik Avenue – Avenue Večeslav Holjevac and the intersection between Savska Road – City of Vukovar Street were selected.

2. OPTIMISATION OF TRAFFIC SIGNAL CYCLE

Traffic management at the intersection will be analysed. A set of movements consists of all the possible streams through an intersection. Sets of movement occur periodically during the control cycle C. The streams and their order of operation during the cycle C have to meet the following conditions:

1. *During cycle C all streams $i \in I$ have to get free passage (green time) at least once.*
2. *Every set has to define the traffic flow at the intersection with no crossing points (points of conflict).*
3. *The change from one set of movements to the other includes "interstage" period which allows vehicles that up to that moment had green light to depart from the intersection and announce free passage for the next stream of vehicles.*

Graph theory can be successfully applied in the interpretation of a set of movements and their order of operation during cycle C, which meet the given conditions. Therefore, two graphs, graph G and graph T are defined. According to their structure, these graphs are connected, non-directional graphs without loops and multiple edges. Graph T is also a complete graph, i.e. every pair of vertices is connected by an edge.

Every element of set I defines one vertex of graph G. The edges of graph G connect movements which have no conflict points. Vertices k and j are connected by a non-directional edge if movements k and j have no common points of conflict. If there is a conflict point, vertices k and j are not interconnected. Graph G can be defined by incidence matrix $A = [\, a_{kj} \,]$ in the following way:

$$a_{kj} = \begin{Bmatrix} 1, \text{ if movements } k \text{ and } j \text{ do not intersect} \\ 0, \text{ if movements } k \text{ and } j \text{ intersect} \end{Bmatrix}$$

Matrix A is a square matrix of type $n \times n$.

Let H be a set of all the complete subgraphs of graph G:

$$H = \{H_a, H_b, \ldots, H_p\}$$

Subgraph of a certain graph $G = (V, R)$ is a new graph G_1 of the form:

$$G_1 = (W, R \cap (W \times W)$$

where $W \subseteq V$.

This means that subgraph of a certain graph is obtained if we isolate certain vertices and those graph edges that connect them. In case of a complete subgraph, each pair of vertices is connected by an edge.

The set of vertices $I_k \subset I$ of subgraph $H_k \in H$ represents a set of movements at a given intersection that can occur simultaneously, and, as already mentioned, with no crossing points. In this way any subgraph from set H represents a set of movements that satisfies condition 2. Let I_H be a set of indices that mark subgraphs of the set H.

$$I_H = \{a, b, c, \ldots, p\}$$

The following correspondence is accepted. Element $f \in I_H$ corresponds to element $H_f \in H$. Set I_H determines the vertices of graph T. Graph T is a complete graph. The edge which connects vertex f with vertex g of graph T is assigned number $S_{fg} = S_{gf}$ that represents the number of common vertices of subgraph H_g and subgraph H_f. ($S_{gf} = S_{fg}$ is the number of elements in the set $I_g \cap I_f$).

Let the closed path q in graph T pass through vertices determined by set I_q ($I_q \subset I_H$), where

$$\bigcup_{i \in I_q} I_i = I$$

If the sets of movements during cycle C are determined by vertices through which path q is passing, then apart from satisfying condition 2, this set also satisfies condition 1.

The order of the set operation during cycle C is determined in the following way. One vertex (any) on path q is assumed to be the initial one. It determines the initial set of movements during cycle C. The order of other sets during the cycle C is determined by the order of the vertices on the path q when the path q is passed by clockwise (or anti-clockwise).

In the periodic operation order of a set of movements, the intersection efficiency is the greater, the lower the number of sets within a cycle, and "overlapping" between sets greater. "Overlapping" between set g and f is so much greater as the number of common vertices of subgraph H_g and subgraph H_f, is greater, i.e. as much as the number $S_{gf} = S_{fg}$ in graph T is greater. Consequently, the optimal number of sets and their order of operation within a cycle, that satisfy conditions 1, 2, 3, can be formulated in the following way:

Closed path q in graph T for which the number of elements in set I_q is minimal, and where

$$\bigcup_{i \in I_q} I_i = I$$

and which gives the function

$$F = \sum_q S_{gx}$$

maximal value, determines the optimal number of sets of movements and their order of operation within a cycle.

This consideration does not take into account the "saturation" of the intersection.

This theory of intersection analysis has been applied to two existing intersections in the city of Zagreb.

3. TRAFFIC SIGNALLING CYCLE OPTIMISATION AT A GIVEN INTERSECTION

Quite a complex intersection in Zagreb has been selected, namely the one at the Dubrovnik Avenue and Avenue Večeslava Holjevca. On the average, 61 traffic accidents per year occur at this intersection. Another characteristic is that all the sets of movements are of equal duration.

Figure 1 shows all the traffic flows at the intersection between Dubrovnik Avenue and Avenue Večeslava Holjevca.

We can write the matrix of incidence A, with elements a_{ij}, where:

$$a_{kj} = \begin{cases} 1, & \text{if movements } k \text{ and } j \text{ do not intersect} \\ 0, & \text{if movements } k \text{ and } j \text{ intersect} \end{cases}$$

$$A = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \\ 7 \\ 8 \\ 9 \\ 10 \\ 11 \\ 12 \\ 13 \\ 14 \end{array} \begin{array}{c} \begin{array}{cccccccccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 & 14 \end{array} \\ \begin{bmatrix} 0 & 1 & 0 & 1 & 1 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 & 1 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 0 & 1 & 1 & 0 \\ 1 & 1 & 0 & 0 & 1 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 & 1 & 1 & 1 & 1 & 0 & 1 & 0 & 0 \end{bmatrix} \end{array}$$

Figure 2 presents graph G, where vertices are the possible streams.

Complete subgraphs of graph G are presented in Figure 3.

Graph T is presented in Figure 4.

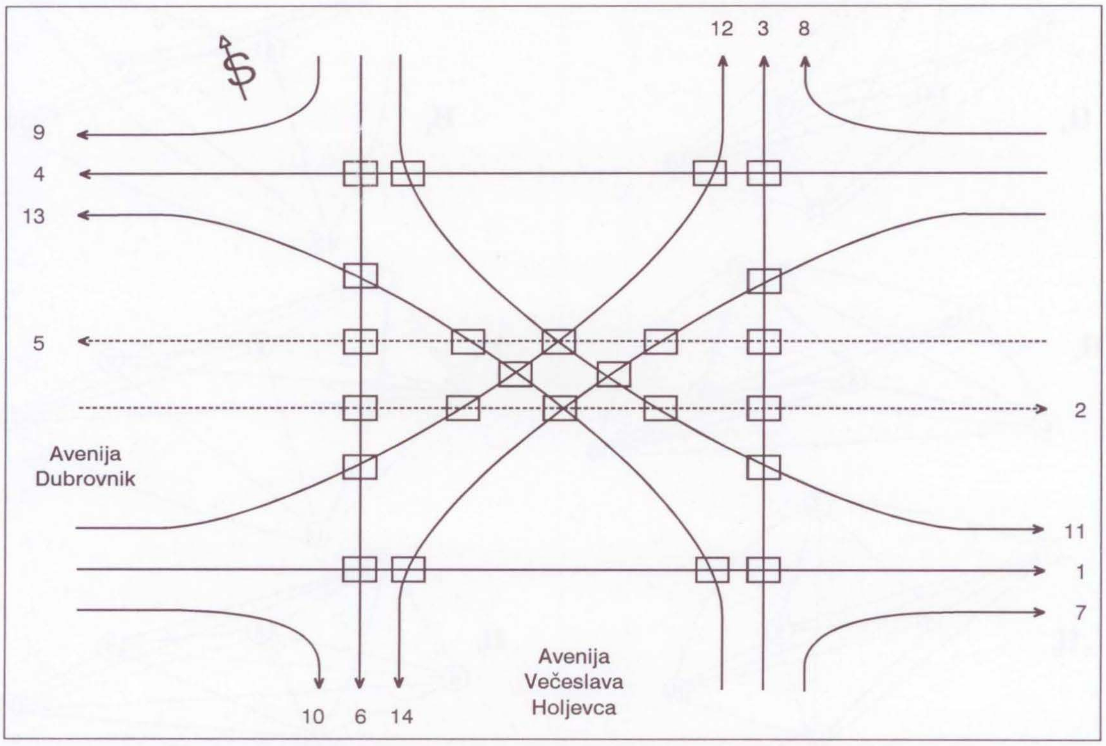

Figure 1 - Traffic flows at the intersection between Dubrovnik Avenue and Avenue Večeslava Holjevca in Zagreb. Streams 2 and 5 are tramways.

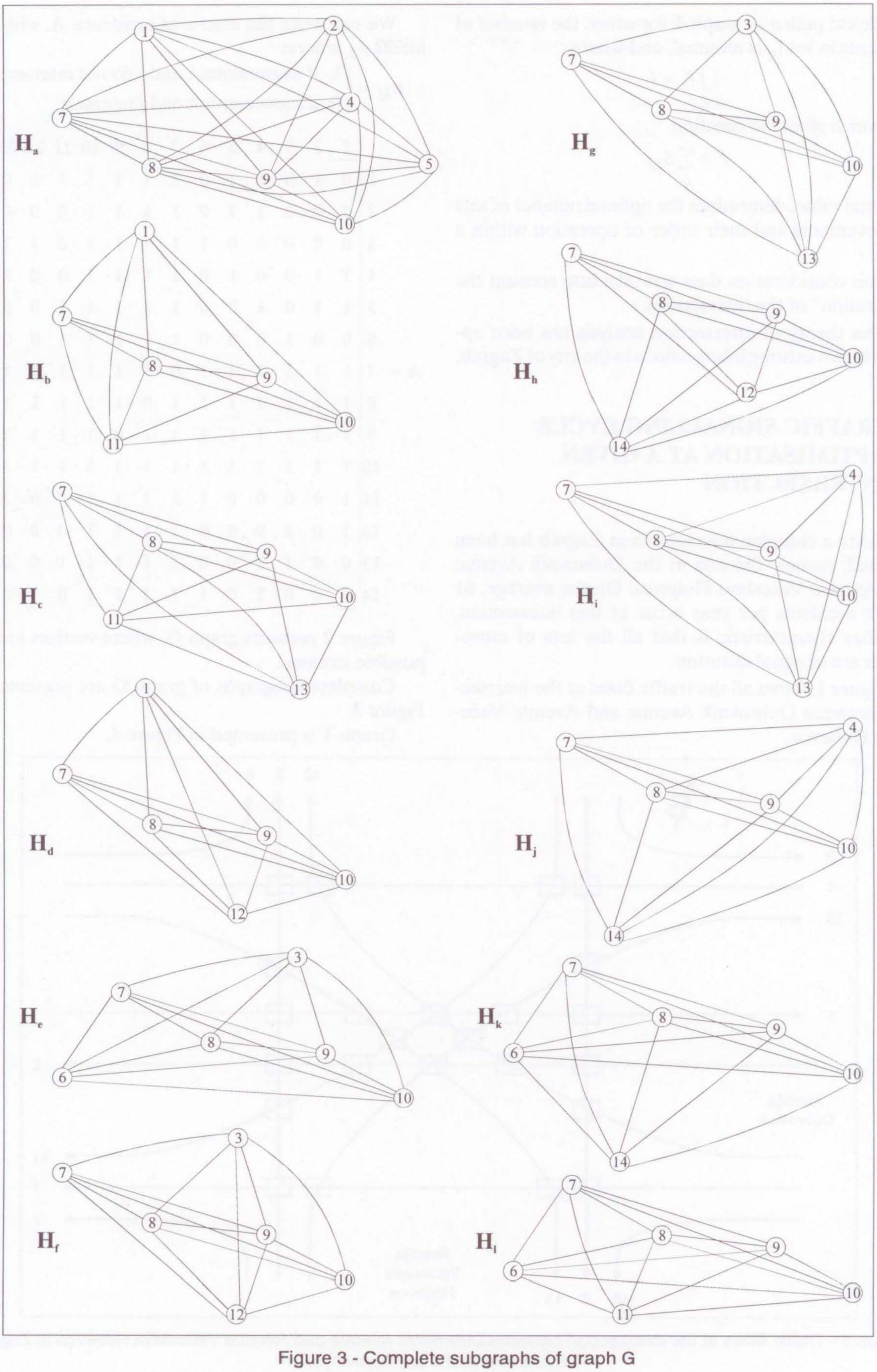

Figure 3 - Complete subgraphs of graph G

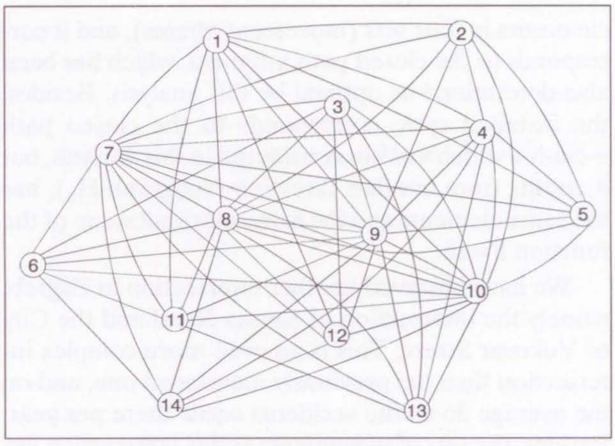

Figure 2 - Graph G

By the analysis of closed paths of graph T the closed path with minimal set is selected, and with maximal value of the criteria functions these are the paths:

1. a-c-e-h-a F=16
2. a-c-h-e-a F=16
3. a-c-f-k-a F=16
4. a-c-k-f-a F=16
5. a-e-c-h-a F=16
6. a-e-h-c-a F=16
7. a-f-c-k-a F=16
8. a-f-k-c-a F=16
9. a-g-h-l-a F=16
10. a-g-l-h-a F=16
11. a-h-c-e-a F=16
12. a-h-e-c-a F=16
13. a-h-g-l-a F=16
14. a-h-l-g-a F=16
15. a-k-c-f-a F=16
16. a-k-f-c-a F=16

Since with periodical order of the sets of movements (traffic signal phases) the efficiency of the intersection is greater if the number of sets within a cycle is lower, the given 16 paths represent potential optimal solutions to the problem. Considering that the order of sets within a cycle is determined anticlockwise (could be clockwise as well), in this case the optimal solution in the cycle is the path (set of movements) a-h-e-c-a under number 12.

By checking and comparing in situ, at the observed intersection it can be concluded that the installed cy-

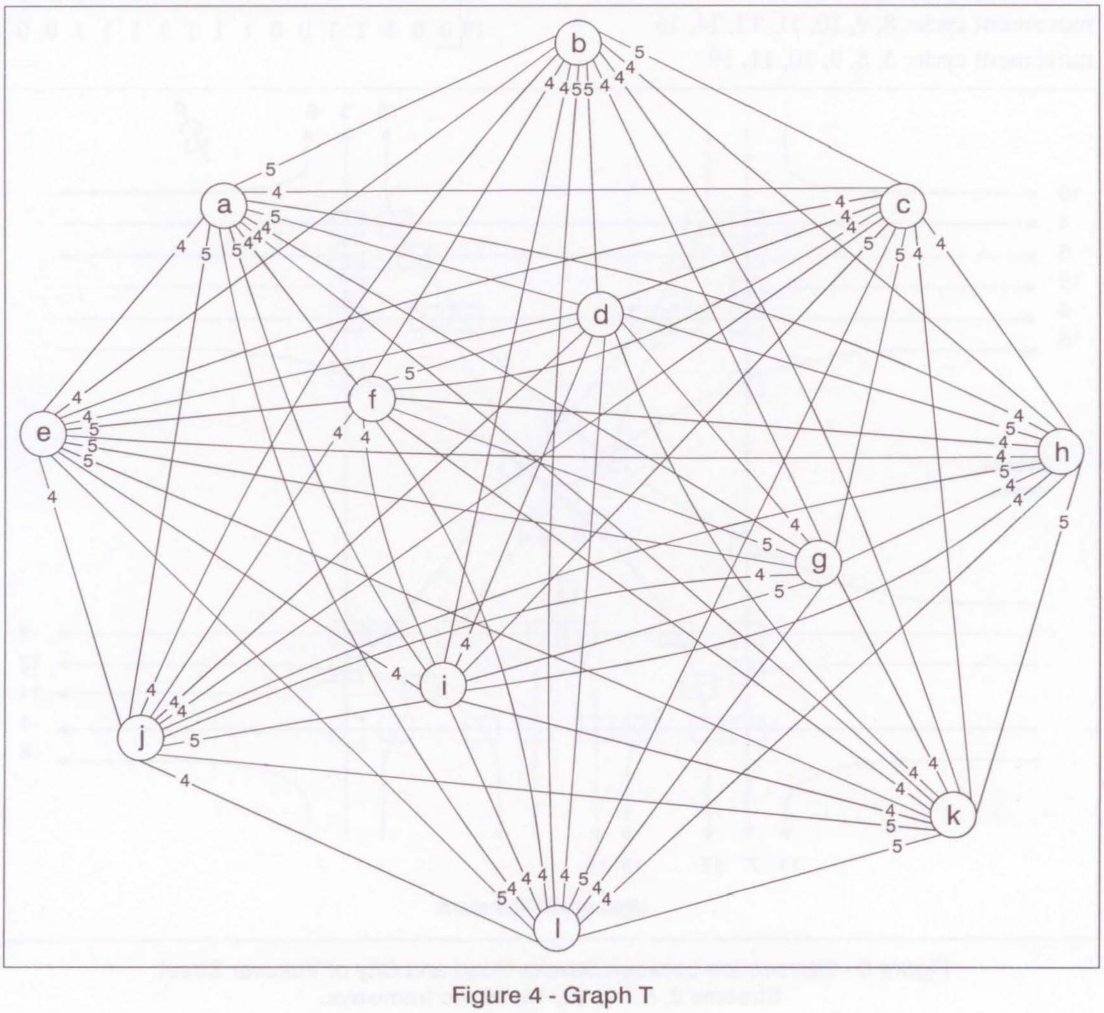

Figure 4 - Graph T

cle occurs in four sets (movement phases), and it corresponds to the closed path a-h-e-c-a, which has been also determined as optimal by this analysis. Besides, the installed cycle corresponds to the closed path e-c-a-h-e which was not considered in this analysis, but it results from another direction (subgraph H_e), has the same elements and the same maximal value of the function F=16.

We have observed another intersection in Zagreb, namely the intersection of Savska Road and the City of Vukovar Street. This is an even more complex intersection than the previously mentioned one, and on the average 36 traffic accidents occur there per year. Besides, the sets of movements at this intersection are not of equal duration.

Figure 5 presents the intersection of the City of Vukovar Street and Savska Road. There are 19 possible elements of movements at this intersection out of which numbers 2, 5, 12, 13, 18, 19 are tramways.

Analysing graph G, complete subgraphs of graph G and graph T we obtained the optimal set:

1st movement cycle: 1, 2, 4, 5, 6, 8, 9, 10, 11

2nd movement cycle: 7, 8, 9, 10, 11, 12, 17

3rd movement cycle: 8, 9, 10, 11, 12, 15, 18

4th movement cycle: 8, 9, 10, 11, 13, 14, 16

5th movement cycle: 3, 8, 9, 10, 11, 19

Incidence matrix A is:

$$
A=\begin{array}{cc}
 & \begin{array}{ccccccccccccccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 & 14 & 15 & 16 & 17 & 18 & 19 \end{array} \\
\begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \\ 7 \\ 8 \\ 9 \\ 10 \\ 11 \\ 12 \\ 13 \\ 14 \\ 15 \\ 16 \\ 17 \\ 18 \\ 19 \end{array} &
\begin{bmatrix}
0 & 1 & 0 & 1 & 1 & 1 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 1 & 1 & 1 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 0 & 1 & 0 & 1 & 1 & 0 & 0 & 1 \\
1 & 1 & 0 & 0 & 1 & 1 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 1 & 1 & 1 & 1 \\
1 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 1 & 1 & 1 & 1 \\
1 & 1 & 0 & 1 & 1 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 0 & 1 & 0 & 0 & 1 & 1 & 0 \\
1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
0 & 1 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 1 & 1 \\
0 & 1 & 1 & 1 & 1 & 1 & 0 & 1 & 1 & 1 & 1 & 1 & 0 & 1 & 1 & 1 & 0 & 1 & 1 \\
1 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 \\
1 & 1 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 1 & 1 & 0 \\
0 & 0 & 1 & 1 & 1 & 1 & 0 & 1 & 1 & 1 & 1 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 1 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 1 & 1 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}
\end{array}
$$

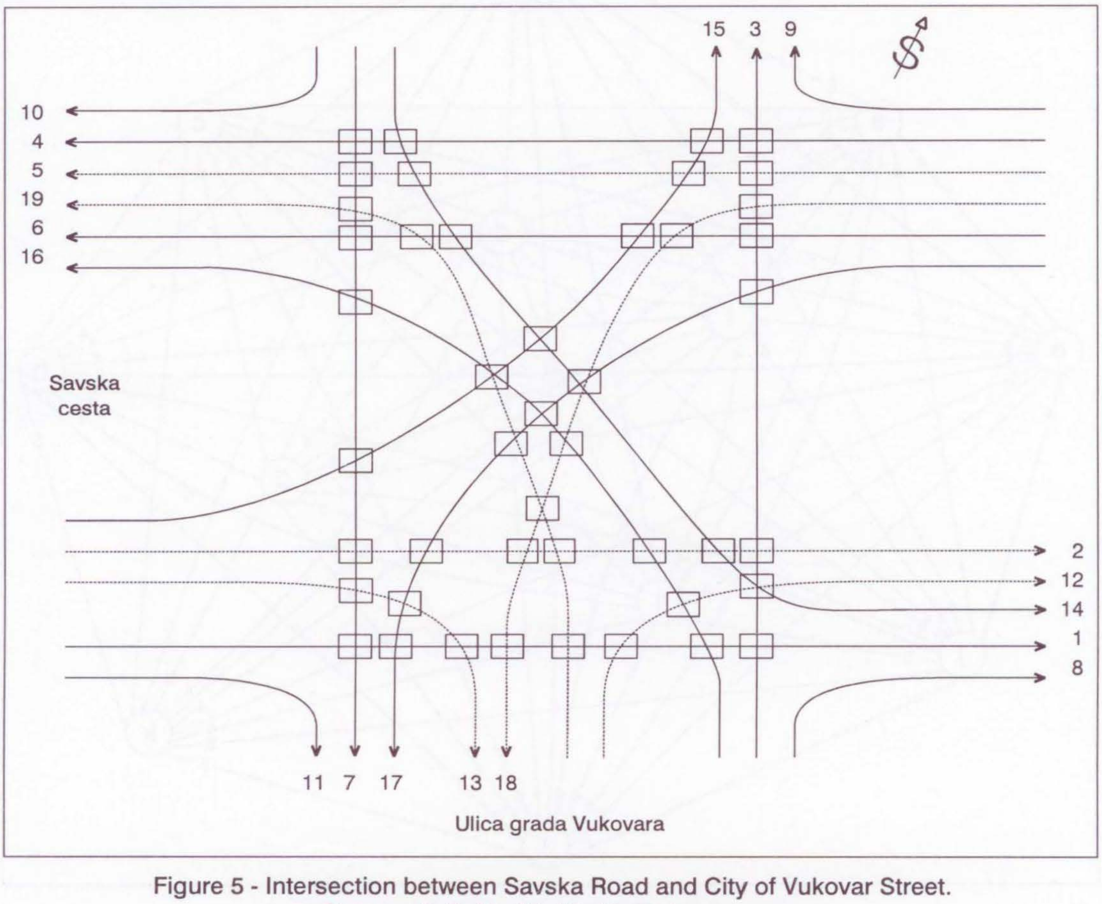

Figure 5 - Intersection between Savska Road and City of Vukovar Street. Streams 2, 5, 12, 13, 18, 19 are tramways.

with maximal value of function F=23.

In the field, the traffic signal setting is somewhat different from the solution that we have obtained by the graph theory, since the second cycle is somewhat shortened.

4. CONCLUSION

Many physically distributed objects (systems) have a network structure. These include: communication systems, road networks, information systems, computer networks and similar systems. This class of systems also includes railway networks, air routes, telecommunication systems, electro-power systems, oil pipe systems, gas pipe systems, water supply and sewage networks, etc.

Common characteristic of all these systems is that objects that form these systems can generally be divided into two classes: terminals and transit points. Depending on the physical properties of the system, connecting lines, edges can have material structure such as e.g. oil pipes, railway lines, maritime routes, radio connections, etc. Regardless of the great diversity, all these systems have numerous common characteristics. For the mathematical description of these systems, that are very different regarding their physical characteristics and regarding the structure of connections, the most suitable are mathematical models which rely on the graph theory.

Solving of the problem regarding the selection of the optimal traffic signal organisation at an intersection using graph theory has been confirmed in this paper at two selected intersections in Zagreb.

At the intersection between Dubrovnik Avenue and the Avenue Večeslava Holjevca, the existing installed organisation of the traffic signal setting is optimal.

At the intersection between Savska Road and the City of Vukovar Street the installed traffic signalling cycle is a somewhat slightly modified optimal organisation. Thus the validity of the method of selecting the optimal traffic signal cycle using graph theory has been confirmed in practice.

SAŽETAK

IZBOR OPTIMALNOG CIKLUSA SEMAFORA POMOĆU GRAFA

Promatra se upravljanje prometom na raskrižju sa semaforom. Jedan ustroj kretanja tvore svi mogući pravci kretanja na raskrižju koji se odvijaju istovremeno. Ustroj kretanja mijenja se periodično tijekom ciklusa upravljanja. U toku ciklusa sva kretanja moraju bar jednom dobiti slobodan prolaz, zeleni signal. Svaki ustroj mora definirati prometni tok na raskrižju bez točaka ukrštanja (konfliktne točke). Pri periodičnom redoslijedu djelovanja ustroja kretanja efikasnost rada raskrižja utoliko je veća koliko je broj ustroja tijekom ciklusa manji, a "preklapanje" između ustroja veće. Analizirana su, pomoću teorije grafova, dva raskrižja u gradu Zagrebu: Avenija Dubrovnik – Avenija Večeslava Holjevca (61 prometna nezgoda na godinu), te raskrižje Savska cesta – Ulica grada Vukovara (36 prometnih nezgoda na godinu)

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